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Hybrid FEM-BEM approach for open boundary magnetostatic problems

Two- and three-dimensional formulations

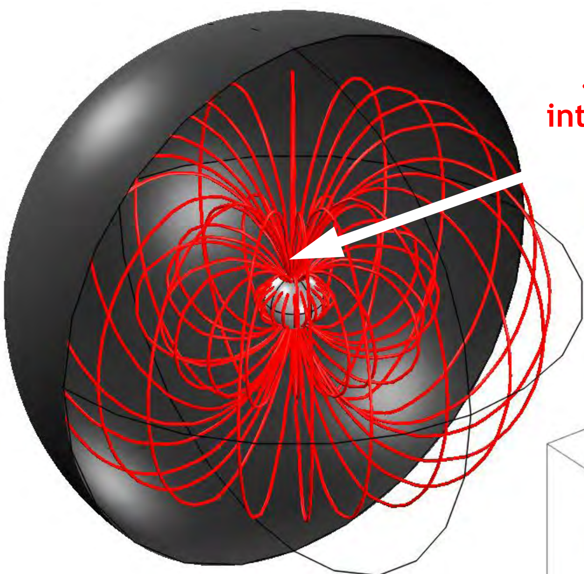
rle AT MIT

Bielefeld University

Motivation

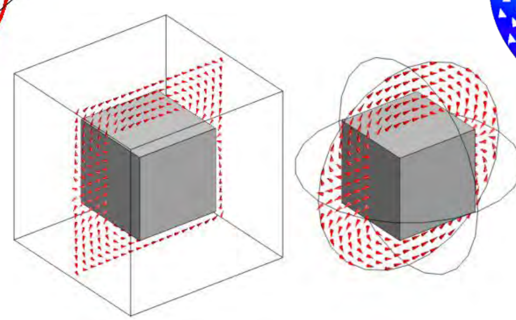
Have ever experienced this?

You have some electromagnetic problem …

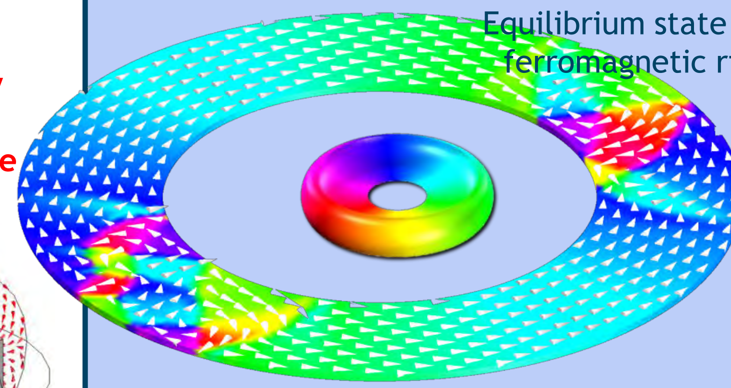


… and the area of interest is somewhere there …

… or you do not know how to choose the exterior domain in order to keep the additional error small?



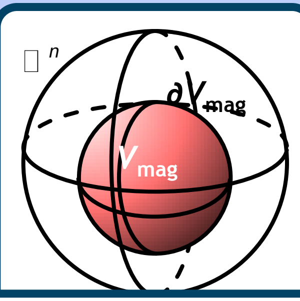
This may be fine as long as we are only interested in the properties of the electromagnetic field. But usually these equations form only a small part of a much larger system.



Equilibrium state of a ferromagnetic ring

Is it possible to eliminate the degrees of freedom in the auxiliary domain?

Open boundary approaches



Maxwell's equations of magnetostatics:

$$\nabla \cdot \boldsymbol{B} = 0 \qquad \boldsymbol{B} = -\mu_0(\boldsymbol{M} + \boldsymbol{H})$$

$$\nabla \times \boldsymbol{H} = 0 \quad \rightarrow \quad \boldsymbol{H} = -\nabla \varphi$$

$$\rightarrow \quad \Delta \varphi = \nabla \cdot \boldsymbol{M} \qquad \forall \boldsymbol{r} \in \mathbb{R}^n$$

$$\tfrac{\partial}{\partial \hat{\boldsymbol{n}}} \varphi = \hat{\boldsymbol{n}} \cdot \boldsymbol{M} \qquad \forall \boldsymbol{r} \in \partial V_{\mathrm{mag}}$$

Greens function G depends on the system dimension n:

$$G(\boldsymbol{r},\boldsymbol{r}') = \frac{1}{2\pi} \ln(|\boldsymbol{r} - \boldsymbol{r}'|)$$ **n = 2**

$$G(\boldsymbol{r},\boldsymbol{r}') = \frac{1}{4\pi} \frac{1}{|\boldsymbol{r} - \boldsymbol{r}'|}$$ **n = 3**

Let's try the decomposition $\varphi = \varphi_1 + \varphi_2$ with

$$\Delta \varphi_1 = \nabla \cdot \boldsymbol{M} \qquad \forall \boldsymbol{r} \in V_{\mathrm{mag}}$$

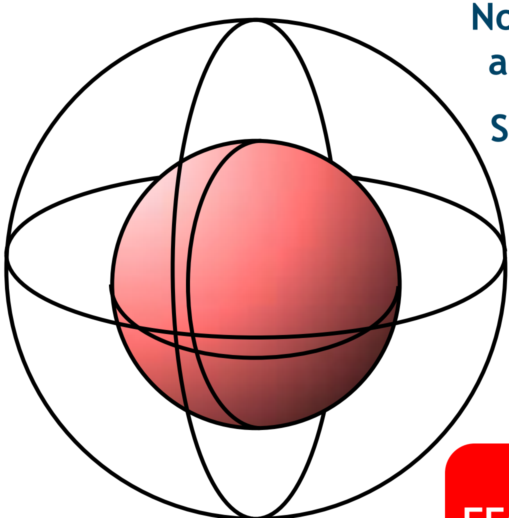
$$\tfrac{\partial}{\partial \hat{\boldsymbol{n}}} \varphi_1 = \hat{\boldsymbol{n}} \cdot \boldsymbol{M} \qquad \forall \boldsymbol{r} \in \partial V_{\mathrm{mag}}$$

$$\varphi_1 = 0 \qquad \forall \boldsymbol{r} \in \mathbb{R}^n \setminus V_{\mathrm{mag}}$$

$$\Delta \varphi_2 = 0 \qquad \forall \boldsymbol{r} \in \mathbb{R}^n$$

$$\varphi_2(\boldsymbol{r}) = \int_{\partial V_{\mathrm{mag}}} \varphi_1(\boldsymbol{r}') \frac{\partial}{\partial \hat{\boldsymbol{n}}} G(\boldsymbol{r},\boldsymbol{r}') d\boldsymbol{r}' + \left(\tfrac{1}{4\pi} \Omega(\boldsymbol{r}) - 1\right) \varphi_1 \qquad \forall \boldsymbol{r} \in \mathbb{R}^n$$

Open boundary approaches



No longer need for auxiliary domain

Solution is exact

hybrid FEM-BEM approach

```
fem.weak{1} = test(phi1x)*(phi1x - Mx) ...
    test(phi1y)*(phi1y - My) ...
    test(phi1z)*(phi1z - Mz);
```

φ_1-mode

```
% Integration coupling variables

expr{1} = -phi1/(4*pi)* ...
    (sign(x)*abs(nx)*(dest(x)-x) ...
    +sign(y)*abs(ny)*(dest(y)-y) ...
    +sign(z)*abs(nz)*(dest(z)-z))*( ...
    (dest(x)-x)^2+ ...
    (dest(y)-y)^2+ ...
    (dest(z)-z)^2)^(-3/2);
```

φ_2-mode

$$\Delta\varphi_1 = \nabla\cdot\boldsymbol{M} \qquad \forall \boldsymbol{r} \in V_{mag}$$

$$\tfrac{\partial}{\partial\hat{\boldsymbol{n}}}\varphi_1 = \hat{\boldsymbol{n}}\cdot\boldsymbol{M} \qquad \forall \boldsymbol{r} \in \partial V_{mag}$$

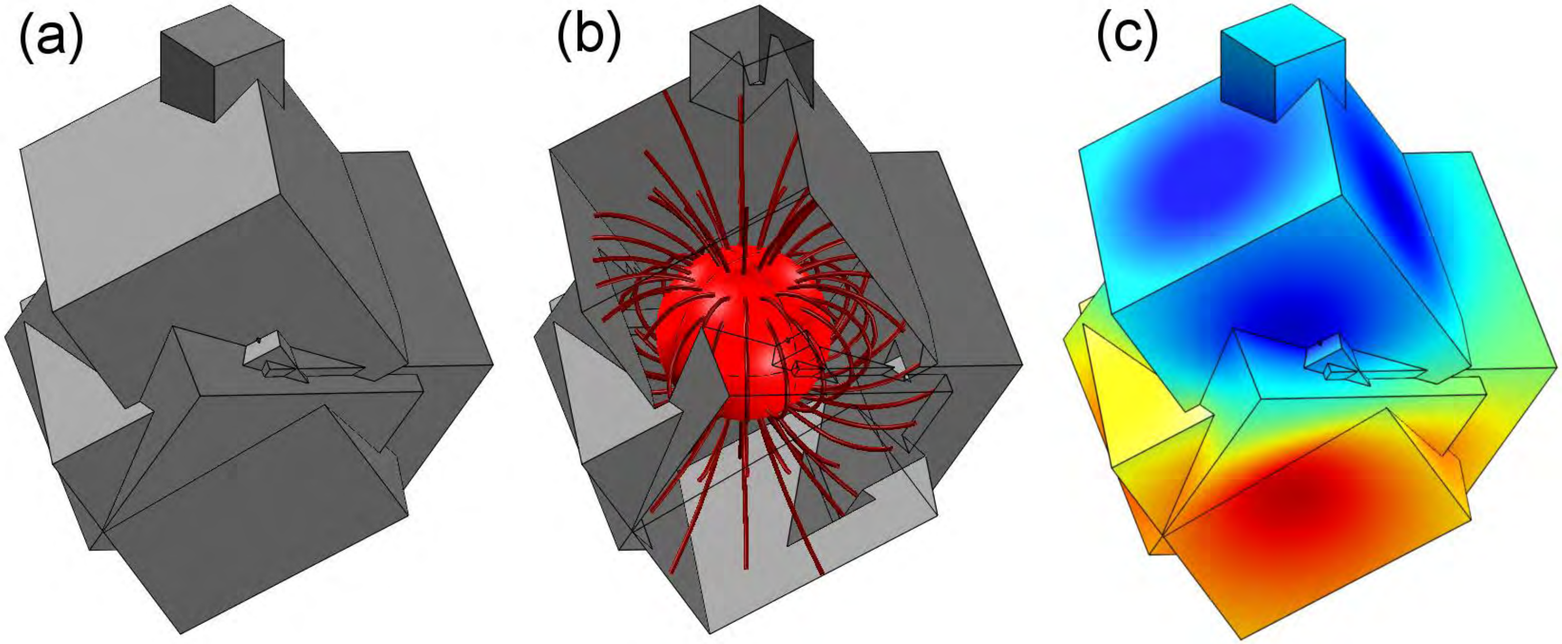
$$\varphi_1 = 0 \qquad \forall \boldsymbol{r} \in \mathbb{R}^n \setminus V_{mag}$$

$$\Delta\varphi_2 = 0 \qquad \forall \boldsymbol{r} \in \mathbb{R}^n V_{mag}$$

$$\varphi_2(\boldsymbol{r}) = \int_{\partial V_{mag}} \varphi_1(\boldsymbol{r}')\frac{\partial}{\partial\hat{\boldsymbol{n}}}G(\boldsymbol{r},\boldsymbol{r}')d\boldsymbol{r}' + \left(\tfrac{1}{4\pi}\Omega(\boldsymbol{r})-1\right)\varphi_1 \qquad \forall \boldsymbol{r} \in \partial V_{mag}$$

Benchmark model

Domain | **Solution** | **Dirichlet data**



Arbitrary exterior domain with angular surface …

… that does not affect the solution of the magnetic stray field …

… due to an appropriate choice of Dirichlet boundary conditions.

Benchmark model

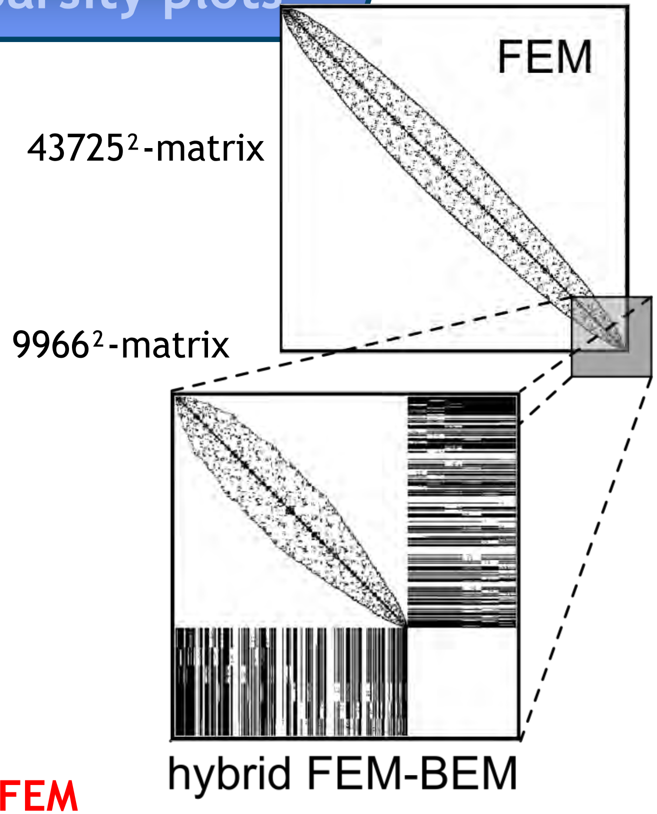
Error analysis

$$\Delta H^1(\varphi_h) = |\varphi - \varphi_h|_{H^1(V_{\mathrm{mag}})}$$
$$= \left(\int_{V_{\mathrm{mag}}} |\nabla\varphi - \nabla\varphi_h|^2 \, \mathrm{d}\boldsymbol{r} \right)^{-1/2}$$

#Elem.	#Bnd. Elem.	$\Delta H^1(\varphi_h)$	t_{sol} [s]
97	80	0.0757	1.466
324	152	0.0470	1.981
1251	320	0.0324	3.113
9098	1240	0.0230	17.191
18119	3495	0.0036	34.725

> 40000 0.0079 3.2 pure FEM

Sparsity plots

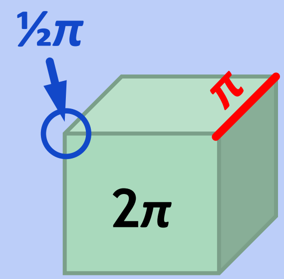


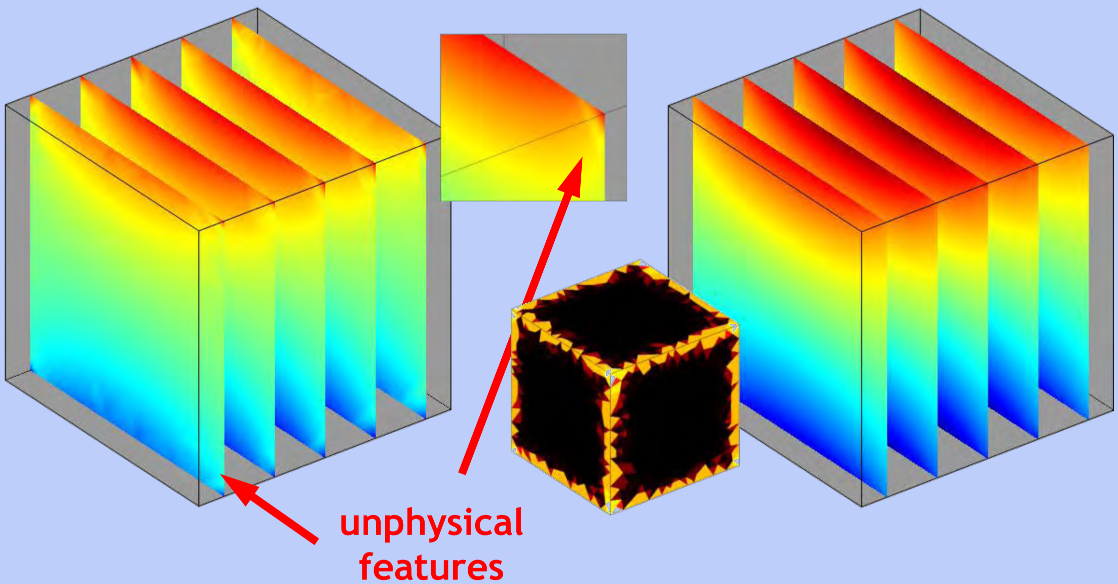
Influence of boundary topology

Homogeneously magnetized cube

$$\varphi_2(\boldsymbol{r}) = \int_{\partial V_{mag}} \varphi_1(\boldsymbol{r}') \frac{\partial}{\partial \hat{\boldsymbol{n}}} G(\boldsymbol{r},\boldsymbol{r}') d\boldsymbol{r}' + \left(\tfrac{1}{4\pi}\Omega(\boldsymbol{r}) - 1\right)\varphi_1$$

$$\Omega(\boldsymbol{r}) = \begin{cases} 4\pi & \text{inner point} \\ 2\pi & \text{along smooth surface} \end{cases}$$





Presolving step

```
...

% Solid angle, Boundaries
fem.appl{3}.mode = FlPDEWBoundary;

...    % use values along edges as
       % boundary values

% Solid angle, Edges
fem.appl{4}.mode = FlPDEWEdge;

...    % use values at corners as
       % boundary values
```

High aspect geometries

Weak equations for finite element discretization

High aspect ratio geometries:

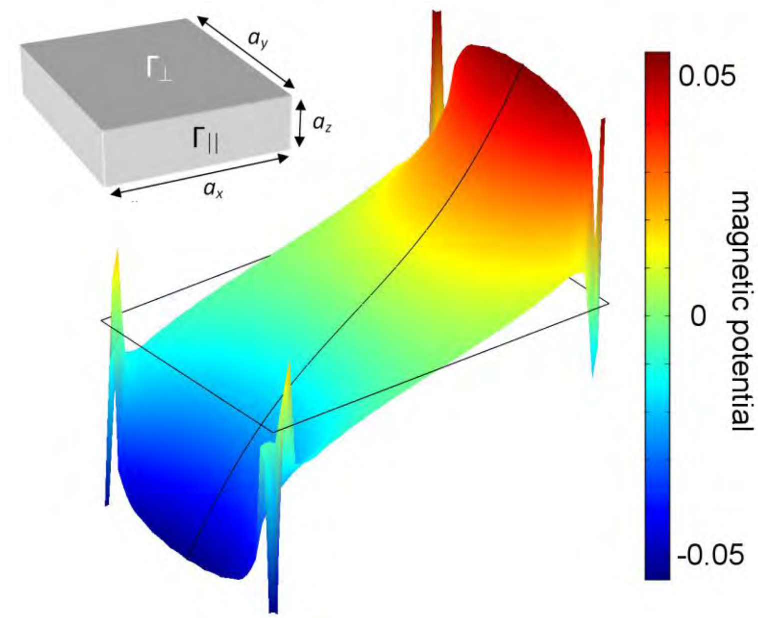
$$\varphi_1(\boldsymbol{r}) \approx \varphi_1(x,y)$$

Integral decomposition

$$\int_{\Gamma_\perp} \varphi_1 \frac{\partial G}{\partial \hat{\boldsymbol{n}}} d\boldsymbol{r}' = -\frac{a_z}{4\pi}\int_{\Gamma_\perp} \varphi_1 \frac{\varphi_1 dx' dy'}{(\Delta \boldsymbol{r}_{xy}^2 + a_z^2/4)^{3/2}}$$

$$\int_{\Gamma_{||}} \varphi_1 \frac{\partial G}{\partial \hat{\boldsymbol{n}}} d\boldsymbol{r}' = -\frac{a_z}{4\pi}\int_{\Gamma_{||}} \varphi_1 \frac{\hat{\boldsymbol{n}} \cdot \Delta \boldsymbol{r}_{xy}}{|\Delta \boldsymbol{r}_{xy}|^3} d\boldsymbol{r}'$$

$$\Delta \boldsymbol{r}_{xy} = (x - x')\hat{\boldsymbol{x}} + (y - y')\hat{\boldsymbol{y}}$$



$a_x = 1.2$, $a_y = 0.8$, $a_z = 0.1$

FEM-BEM: 0.17 s

FEM: 1.72 s

Increased performance due to reduced dimensionality, thickness only enters as numerical parameter.

Conclusion & Outlook

Conclusion

- Hybrid FEM-BEM approaches can be implemented into COMSOL Multiphysics
- So far, for full three-dimensional problems, no increase of performance can be reported due to decreased sparsity of the matrix
- Increased performance for geometries of high aspect ratios

Outlook

- Further optimization of solver parameters and settings
- Implementation of hybrid FEM-BEM methods in ferromagnetic systems