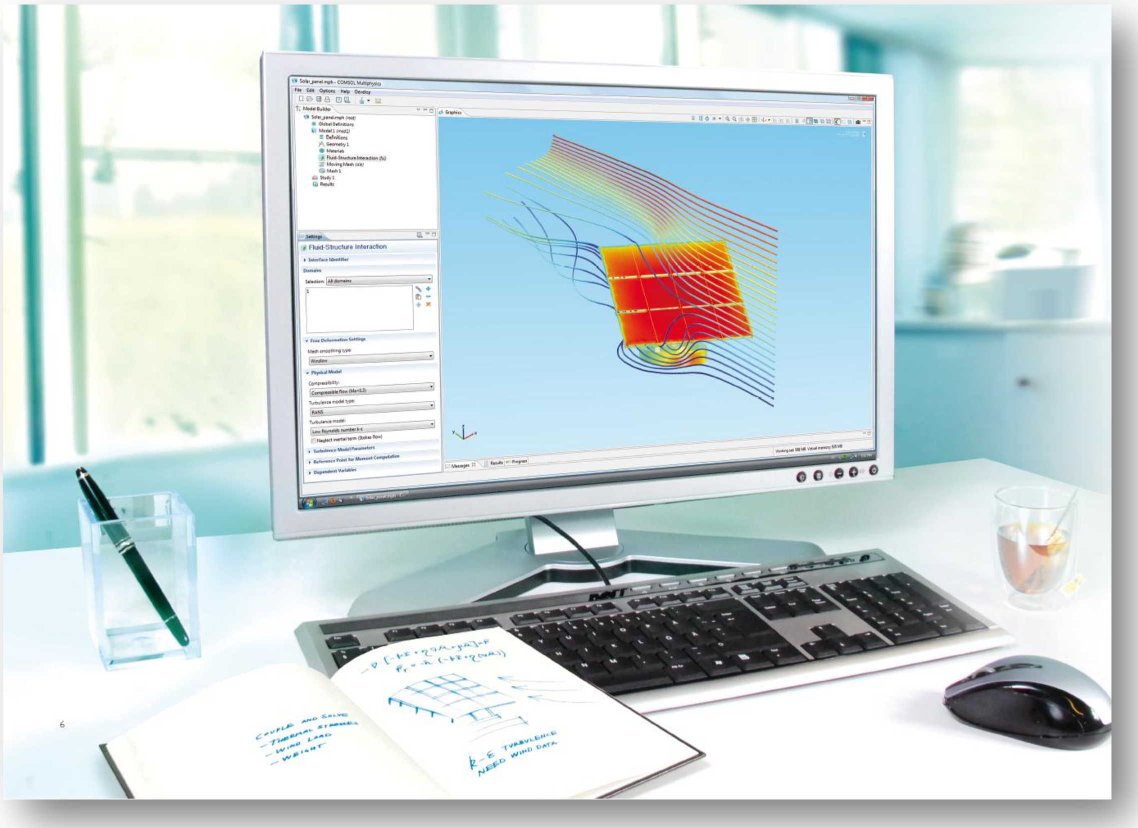
COMSOL

Vill du vara med i vårt multifysiklag?



Vi söker dig som vill göra examensarbete inom numeriska metoder för tillämpningar inom flödesdynamik. Arbetet utförs på COMSOL och på Institutionen för Matematik.

Din profil

- Nogrann, metodisk och kreativ.
- Bra på att skapa strukturerad och effektiv kod.
- Intresserad av modellering och simulering.

Vill du veta mer?
Kontakta Ed Fontes, ed.jobb@comsol.se, eller Henrik Shahgholian, henriksh@math.kth.se.

Välkommen med en ansökan via e-post till ed.jobb@comsol.se *eller till COMSOL AB, Tegnérgatan 23, 111 40 Stockholm*

NUMERICAL APPROXIMATION FOR A MULTI PHASE FREE BOUNDARY PROBLEM

Beyond question, the theory of free boundary problems, FBPs, has seen great progress along with a vast of applications in physics, industry, finance, biology, and other areas. The corresponding governing partial differential equations of FBPs exhibit apriori unknown sets, free boundaries, such as interfaces, moving boundaries, shocks, etc. However the mathematical and relevant literature of this field is enormous, at this point we refer to [10] to review a number of applications of FBPs in science and industry.

1. Problem Setting

Let μ_i, $1 \leq i \leq k$ be finite measures with compact supports and $\lambda_i(x)$ be non-negative Lipschitz continuous functions.

Problem: Find non-negative functions u_i and the corresponding domains $\Omega_i := \{u_i > 0\}$ for $1 \leq i \leq k$ such that $\operatorname{supp}(\mu_i) \subset \Omega_i$ and

$$(1) \qquad \begin{cases} \Delta u_i = \lambda_i \chi_{\Omega_i} - \mu_i & \text{in } \Omega_i, \\ u_i = 0 & \text{on } \partial\Omega_i, \\ |\nabla u_i| = |\nabla u_j| & \text{on } \Gamma_{ij} := \partial\Omega_i \cap \partial\Omega_j, \\ |\nabla u_i| = 0 & \text{on } \partial\Omega_i \setminus \bigcup \Gamma_{ij}, \end{cases}$$

The $\Omega := \bigcup_i \Omega_i$ is called a multi-phase quadrature domain for the measure $\mu := \sum_i \mu_i$, see Figure 1.

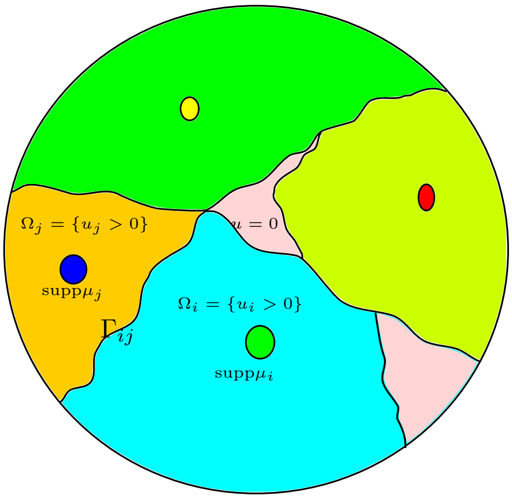


FIGURE 1. We would like to construct a numerical scheme to approximate functions u_i and the domains $\Omega_i = \{u_i > 0\}$ along with all conditions of the problem (1). The boundaries $\partial\Omega_i$ are the free boundaries which are not known a priori.

2. Applications in fluid dynamics

Recent researches have shown that the context of quadrature domain theory can be understood as the solution of specific fluid dynamic problems. Here we will survey a number of physical applications of quadrature domains by emphasizing on the special cases of the problem.

2.1. One phase case

For $k = 1$ the problem arises from an interesting and quite well known theory which is called one phase quadrature domain. While for the mathematical theory one can find vast of literature, it is really remarkable to point out that one phase quadrature domain is relevant to a wide range of physical problems.

Richardson in [15] made the first connection between quadrature domain and fluid mechanics who studied the free boundary arising by the motion of a fluid trapped in a cell. Suppose that an incompressible fluid is confined between two parallel plates, Hele-Shaw cell, and we inject more fluid into it with moderate velocity. Therefore, the fluid between plates will occupy more space. When the surface tension effects on the free boundaries are ignored, this free boundary problem could be understood as the one phase case of the problem (1) by considering an appropriate measure. For a comprehensive review of quadrature domain and the impact of it in the context of flows in porous media and Hele-Shaw flows, we refer the reader to [11], [12], [13], [15], [19] and [21] and references therein.

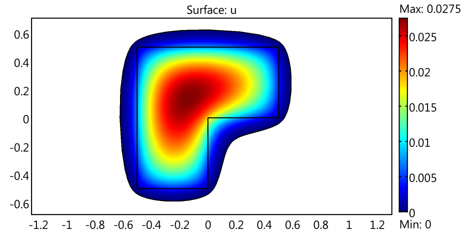


FIGURE 2. A one phase quadrature domain where the polygon is the support of the measure.

2.2. Two phase case

In case $k = 2$, when $\mu_1 = \mu_2 = 0$ the problem could be considered like the two-phase obstacle problem or the two-phase membrane problem. Here the free boundary consist of

$$\partial\{u > 0\} \quad \text{and} \quad \partial\{u < 0\}.$$

One of the challenging issues in this case is that the interface could be considered as two parts. One part is when the gradient of u vanishes and one where the gradient non zero. Then because of these two decompositions of two different types of growth, it is not easy to deduce a growth estimate at points on the interface, see for instance [22] and [20] and references therein.

An application of two phase case in fluid dynamics is the two phase Hele-Shaw flows (Muskat problem) which will be discussed in the next subsection. Another interesting application of two phase obstacle problem is the consideration of a thin film or membrane which is fixed on the boundary of a domain.

Moreover, a part of the membrane is under a thick liquid which is supposed to be heavier than the membrane. Now the membrane produces a pressure downward on the part of the membrane which is above the liquid, say λ^+. On the other hand the part in the liquid is also push up by another force, λ^-, because of the liquid's weight. The mathematical interpretation of the equilibrium state is the two phase obstacle problem.

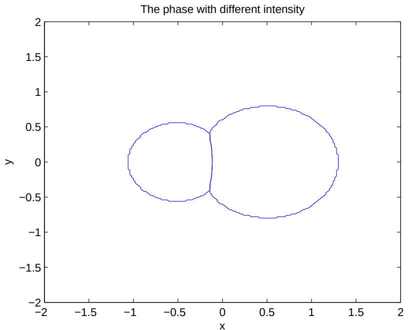


FIGURE 3. A two phase quadrature domain by considering two Dirac measures concentrated on two points.

If one considers the two phase case with the measures $\mu^{\pm}$, the problem turns to the two phase quadrature domains which was introduced in [9] where the authors have proved the existence by considering some restrictions. It is worthwhile to mention that there is no concrete theory for the uniqueness in this case.

2.3. General case

This problem could be translated into multi phase fluid theory which is a generalization of two-phase flow. Here, the phases are non mixable or non chemically related. The importance of multi phase flows is seen in a wide variety of industries, including power, petroleum, and numerous processing industries.

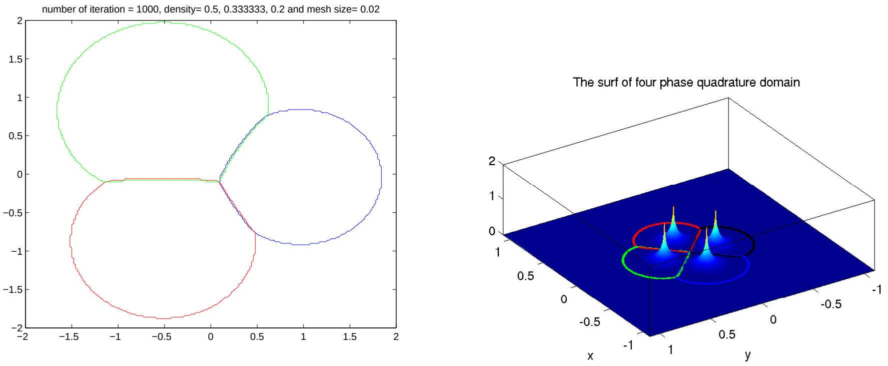
To be more precise, consider an empty Hele-Shaw cell and inject fluids at a number of points into the gap. The fluids regions grow as circular discs and initially we have disjoint blobs of fluids which the growth of any one does not affect by the others. But they will coalesce and will form a multi-phase quadrature domain.

During this process we will have number of holes between the blobs and consequently some interesting questions arise,

- How can we construct an efficient numerical scheme for the problem?
- Do we have any hole when we change the number of injection points?
- What kind of assumptions are sufficient to have a connected multi phase quadrature domain?
- Numerically, how one can estimate the convergence rate of the numerical scheme?

For more information about multi-connected quadrature domains see [7], [8], [5], [16], [17] and [18] and references therein. Indeed, [4] is also a good reference of the application of quadrature domains in fluid dynamics. Figure (4) illustrates the simulation of a three and four phase cases where we have three and four injection points.

From mathematical standpoint, the general case, i.e, multi phase case, is a complicated problem. For instance, while many explicit solutions to the single-phase Hele-Shaw problem are known, solutions to the two-phase problem (also known as the Muskat problem) are scarce. To our best knowledge there is no substantial theory concerning existence, uniqueness and geometrical properties of the solution and the corresponding free boundary. However the more general mathematical question of the global existence and well-posedness of the Muskat problem has also been the focus of an investigations by Siegel, Caflisch and Howison [2] and Ambrose [14].



(A) A three phase quadrature domain where the measures are considered as Dirac which are located on a circle.

(B) The surface of a four phase quadrature domain where the measures are considered as Dirac which are located on a circle.

FIGURE 4. Three and four phase quadrature domains. One can consider the Hele-Shaw problem with three and four injection points.

3. Suggested numerical schemes

3.1. First algorithm

One can prove that if u solves the problem in one phase case then it satisfies

$$L(x,u,Du,D^2u) := \min(-\Delta u + \lambda - \mu, u) = 0, \tag{2}$$

in viscosity scenes, see [1] and [3]. We call it *Min-formula.*

Similarly in two phase case suppose that u is the solution of the problem and let

$$\Omega_1 = \Omega^+ = \{x|u(x) > 0\} \qquad \text{and} \qquad \Omega_2 = \Omega^- = \{x|u(x) < 0\}.$$

Then it is not complicated to prove that u satisfies in the following non linear equation in $\Omega = \Omega_1 \cup \Omega_2$, which is called *Min-Max* formula,

$$Lu := \min\Big(-\Delta u + \lambda^+ - \mu^+, \max(-\Delta u - \lambda^- + \mu^-, u)\Big) = 0. \tag{3}$$

Now we can easily discretize the *Min-formula* and *Min-Max* formulas. Here we just make the algorithm for the latest one.

Algorithm I: The second numerical algorithm based on Min-Max formulation is given.

(1) Choose a tolerance, TOL.
(2) Choose a big domain D and consider a finite mesh on it.
(3) Find an appropriate discretization $\mu_h^{\pm}$ for the measures $\mu^{\pm}$.
(4) By simple computations we can choose

$$u_i^{k+1} = \max\left(\overline{u_i^k} + \frac{\mu_h^+ - \lambda^+}{4}h^2, \min(\overline{u_i^k} + \frac{\lambda^- - \mu_h^-}{4}h^2, 0)\right).$$

(5) If $|u_i^{k+1} - u_i^k| \leq \mathrm{TOL}$ then stop otherwise iterate the step (4).

Indeed, by Barles-Souganidis Theorem we can prove that this scheme is convergent.

3.2. Second algorithm

Now consider a domain D and define a grid on it which consists a set of nods $(x_i, y_i) \in D$. suppose that $\Omega \subset D$ and $\mathcal{N}$ be a uniform mesh on D with the mesh size h. We use the five stencil points finite difference for Laplace operator to get

$$\begin{aligned}\frac{4}{h^2}\left(\overline{u_1}(x_i,y_j) - u_1(x_i,y_j) - \overline{u_2}(x_i,y_j) - u_2(x_i,y_j)\right) &= \\ = (\lambda^+\chi_{\Omega_1} - \mu^+) - (\lambda^-\chi_{\Omega_2} - \mu^-),&\end{aligned} \tag{4}$$

where $\overline{u}(x_i, y_j)$ indicates the average of u on all neighborhood of $(x_i, y_j) \in \mathcal{N}$. We can obtain $u_1(x_i, y_j)$ and $u_2(x_i, y_j)$ from (4) and impose the following conditions

$$u_1(x_i,y_j)\cdot u_2(x_i,y_j) = 0 \text{ and } u_1(x_i,y_j) \geq 0,\ u_2(x_i,y_j) \geq 0.$$

The iteration method is set up as follows,

$$u_1^{k+1}(x_i,y_j) = \max\left(\overline{u_1^k}(x_i,y_j) - \overline{u_2^k}(x_i,y_j) + \frac{(\mu_h^+ - \lambda^+)h^2}{4}, 0\right), \tag{5}$$

and

$$u_2^{k+1}(x_i,y_j) = \max\left(\overline{u_2^k}(x_i,y_j) - \overline{u_1^k}(x_i,y_j) + \frac{(\mu_h^- - \lambda^-)h^2}{4}, 0\right), \tag{6}$$

where $\mu_h^\pm$ are discretization of $\mu^\pm$.

Algorithm II: Here we give the first numerical algorithm based on PDE formulation for two phase case.

1. Choose a big domain D and consider a finite mesh on it.
2. Find an appropriate discretization for the measures $\mu^\pm$.
3. By using (5) and (6) find u_1 and u_2 and iterate this step.

3.3. Multi phase case

Now we describe another method in presence of three phase, which is based on the two phase algorithm. For $i = 1, 2, 3$ suppose that Ω_i are the desirable corresponding quadrature domains for the measures μ_i. In addition, we note that the supports of these measures are disjoint and $\mathrm{supp}(\mu_i) \subset \Omega_i$.

Algorithm III: The algorithm for $j \geq 1$ is constructed as follows.

- Consider just two measures μ_1 and μ_2. Solve the corresponding problem due to the algorithm **III** and find Ω_1^j and Ω_2^j.
- Fix the Ω_1^j and solve the two phase problem for μ_2 and μ_3 in $\mathbb{R}^n \setminus \Omega_1^j$ by considering that the boundary value is vanished on the $\partial\Omega_1^j$. The related domains is called Ω_2^j and Ω_3^j.
- Now fix Ω_2^j and do the first step for μ_1 and μ_3 in $\mathbb{R}^n \setminus \Omega_2^j$ and find Ω_1^{j+1} and Ω_3^{j+1}.

It seems that one is able to construct more numerical methods to approach the solution. Our aim is to extend and improve efficient numerical schemes to investigate the problem.

References

[1] Bardi, M., Crandall M. G., Evans L. C. , Soner H. M. and Souganidis E. P., Viscosity Solutions and Applications, Lecture Notes in Math. 1660, Springer-Verlag, Berlin, 1997.

[2] Caflisch R.E., M. Siegel, S. D. Howison, Global existence, singular solutions and ill-posedness for the Muskat problem, Comm. Pure. Appl. Math. 57 (2004) 1374-1411.

[3] Crandall, Michael G., Ishii, Hitoshi., Lions, Pierre-Louis., User's guide to viscosity solutions of second order partial differential equations, Bull. Amer. Math. Soc. (N.S.) 27 (1992), no. 1, 1-67.

[4] Crowdy, D., Quadrature domains and fluid dynamics. 113-129, Oper. Theory Adv. Appl., 156, Birkhuser, Basel, 2005.

[5] Crowdy, D., Multiple steady bubbles in a Hele-Shaw cell. Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. 465 (2009), no. 2102, 421-435.

[6] Crowdy, D., Exact solutions to the unsteady two-phase Hele-Shaw problem. Quart. J. Mech. Appl. Math. 59 (2006), no. 4, 475485.

[7] Crowdy, D., Marshall J., Multiply connected quadrature domains and the Bergman kernel function. Complex Anal. Oper. Theory 3 (2009), no. 2, 379-397.

[8] Crowdy, D., Mineev-Weinstein M., Putinar, M., Preface [Quadrature domains and Laplacian growth in modern physics]. Complex Anal. Oper. Theory 3 (2009), no. 2, 313-332

[9] Emamizadeh B., Prajapat J., Shahgholian H., A two-phase free boundary problem related to quadrature domains, Potential Anal, Springer Science, 2010.

[10] Friedman, A., Free boundary problems in science and technology. Notices Amer. Math. Soc. 47 (2000), no. 8, 854-861.

[11] Gustafsson B., On quadrature domains and an inverse problem in potential theory, J.Analyse Math. 55 (1990), 172-216.

[12] Gustafsson B., Applications of variational inequalities to a moving boundary problem for Hele Shaw flows, SIAM J. Math. Anal. 16 (1985), 279-300.

[13] Gustafsson B., Shahgholian H., Existence and geometric properties of solutions of a free boundary problem in potential theory, J. Reine Angew. Math. 473 (1996), 137-179.

[14] Ambrose D.M, Well-posedness of the two-phase Hele-Shaw flow without surface tension, Europ. J. Appl. Math. 15 (2005) 597-607.

[15] Richardson S., Hele-Shaw flows with a free boundary produced by the injection of fluid into a narrow channel, J. Fluid Mech., 56 (1972), pp. 609-618.

[16] Richardson, S., Hele-Shaw flows with time-dependent free boundaries involving a multiply-connected fluid region. European J. Appl. Math. 12 (2001), no. 5, 571-599.

[17] Richardson, S., Plane Stokes flows with time-dependent free boundaries in which the fluid occupies a doubly-connected region. European J. Appl. Math. 11 (2000), no. 3, 249-269.

[18] Richardson, S., Hele-Shaw flows with time-dependent free boundaries in which the fluid occupies a multiply-connected region. European J. Appl. Math. 5 (1994), no. 2, 97-122.

[19] Sakai M., Quadrature Domains, Lect. Notes Math. Vol. 934, Berlin-Heidelberg: Springer-Verlag, 1982.

[20] Shahgholian, H., Nina Uraltseva, and Georg S. Weiss, Global solutions of an obstacle-problem-like equation with two phases, Monatsh. Math. 142 (2004), no. 1-2, 27-34.

[21] Varchenko A.n., Etingof P.E, Why the boundary of a round drop becomes a curve of order four, American Mathematical Society University Lecture Series, 3, (1994).

[22] Weiss, G. S., An obstacle-problem-like equation with two phases: pointwise regularity of the solution and an estimate of the Hausdorff dimension of the free boundary, Interfaces Free Bound. 3 (2001), no. 2, 121-128.