A review of the problem
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1 The w-equation

This is just to make sure we're on the same page. I am going to omit the subscript x from the v and y from the k to avoid confusion. We have 3 equations in 3 unknowns, w, v, k

$$\frac{\partial w}{\partial t} = -\frac{\partial}{\partial y}\left(\frac{2\beta k_{x0} k(y,t)}{\left(k_{x0}^2 + k(y,t)^2\right)^2} w(y,t)\right) + \left[\gamma_0 e^{-0.1y} - D_0 e^{.01y}\left(k_{x0}^2 + k(y,t)^2\right)\right] w(y,t) \quad (1)$$

$$\frac{\partial v}{\partial t} = \frac{\partial}{\partial y}\left(\frac{2\beta k_{x0} k(y,t)}{\left(k_{x0}^2 + k(y,t)^2\right)^2} w(y,t)\right) - \nu v(y,t) \quad (2)$$

$$\frac{\partial k}{\partial t} = -k_{x0}\frac{\partial v}{\partial y} \quad (3)$$

$$v_g = \frac{2\beta k_{x0} k(y,t)}{\left(k_{x0}^2 + k(y,t)^2\right)^2} \quad (4)$$

A couple of things about the equations. The forcing (γ) and damping (D) profiles are not fixed. Anything that prevents the solution from reaching and hitting the far boundary will work for the damping. We also need to make sure we force enough not to kill off w entirely. I noticed in my 'Method of Lines' code that the solutions seem to be stable until $[force - damp] < 0$ and that seems to be when it goes unstable. This change in sign happens because k increases in time. In any event, any profiles, provided the forcing is local relative to the damping, that give stable solutions would work. I originally wanted to use a step function (for $force$) to force at a constant value out until, say, $y = 50$ and then have nothing but I think this could lead to numerical issues.

The equations are built so when the w equation is added to the v equation they conserve momentum (up to forcing and damping). This may be an issue since it yields the flux matrix (when written in matrix from, the 3×3 matrix that ∂_y acts on) has a zero eigenvalue. I added an additional term to the v equation in a test case and didn't get different results so I'm not sure if this makes a big difference or not.

I use the following boundary/initial conditions

$$w(0,t) = w_0;\ w(100,t) = 0;\ w(y,0) = w_0 e^{-.1y}$$

$$v(0,t) = v_0;\ v(100,t) = 0;\ v(y,0) = v_0\cos(3y)$$

$$k(0,t) = k_0;\ k(100,t) = 0;\ k(y,0) = k_0 e^{-.05y}$$

The initial w and k conditions are not that important provided they are fairly localized around 0. A delta function would be ideal but again this is numerically an issue. I have tried a Gaussian and achieved essentially the same results. The initial v must have some periodicity as the zonal flow is sheared leading to small w bump propagation. I achieved similar results with $\cos(y)$ and this is fine if it is more stable. I run the following checks on the constants before solving

$$\gamma_0 > (k_{x0}^2 + k_0^2)D_0;\ \gamma_0 > \nu$$

$$\gamma_0 > v_g(t = 0)/L;\ \gamma_0 < \beta/k_{x0}$$

Here L is the box length. Provided these relations are satisfied, the values of the constants are not important. As I mentioned, I was able to get a propagating bump solution (see movie on website) for the w equation with the following constants

$$w_0 = 15; v_0 = 2; k_0 = 5 = k_{x0}; \gamma_0 = 7; \nu = 0.1 = D_0; L = 100; \beta = 80;$$

I also had to the diffusive term in the k equation, meaning I solved

$$\frac{\partial k}{\partial t} = -k_{x0}\frac{\partial v}{\partial y} + D_1\frac{\partial^2 k}{\partial y^2}$$

where the diffusion coefficient was $D_1 = 1$. As I decrease the diffusion coefficient, the solution exhibits more and more oscillatory behavior. PDEPE will not produce a solution when $D_1 = 0$. I should also note that Matlab gives different solutions depending on the ordering of the equations, which makes me uneasy about the results. It at least gives a solution when the k equation is solved first, which is more than my 'Method of Lines' code which is unstable to the max. In any event, I wanted to check the Matlab solution and solve a new system, with only modifications to the w equation

$$\frac{\partial w}{\partial t} = -\frac{\partial}{\partial y}\left(\frac{2\beta k_{x0}k(y,t)}{\left(k_{x0}^2 + k^2\right)^2}w\right) + \left[\gamma_0 e^{-0.1y} - D_0 e^{.01y}\left(k_{x0}^2 + k^2\right)\right]w + \frac{\partial}{\partial y}\left(D_{nl}w\frac{\partial w}{\partial y}\right) - \alpha_{nl}w^2 \quad (5)$$

In this case, $\alpha \sim \gamma_0$ and $D_{nl} \sim D_0$. This version of the equations might be more stable. Matlab gives me a solution, but it is independent of D_{nl}, which seems absurd. I tried $D_{nl} = .01$ and $D_{nl} = 1000$ and it gave the same solution. Matlab does give different solutions for different coefficients if I am only solving the one equation, nonlinear diffusion problem. Any help or insight you can provide is much appreciated. This system has proven quite difficult to handle.