Weakly non-linear analysis of azimuthal thermoacoustic modes in annular combustion chambers

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Thermoacoustic instabilities are **self-excited** pressure oscillations generated by a mutual interaction between heat release fluctuation and pressure waves which develop in the combustion zone.

**Combustion dynamics ➔ Acoustics**

In general, the thermoacoustic oscillations are associated with one natural pure acoustic modes of the combustion chamber of the system (bulk, axial and transverse modes).
Thermoacoustic Instabilities

Krebs et al., *Journal of Engineering for Gas Turbines and Power*, 135 (8), 081 503.
NonLinear Flame Models are able to

- Get the limit cycle amplitude
- say if the system is subject to hysteretic phenomena
- provide information about the range of stability for certain parameters
Objective

- Introduction of non-linearities in the Flame Transfer Function model
- Determine the bifurcation diagrams for these flame models by means of a weakly non-linear analysis.

Develop a simplified 3D FEM tool able to study the nonlinear behavior (limit cycles amplitudes, hysteresis cycles, triggering) of annular combustion chambers.
Mathematical Model

Wave equations with damping and acoustic source

\[
\frac{1}{c^2} \frac{\partial^2 p'}{\partial t^2} + \frac{\zeta}{c R R} \frac{\partial p'}{\partial t} = \frac{\rho}{c^2} \nabla \cdot \left( \frac{1}{\rho} \nabla \hat{p}' \right) = \frac{\gamma}{c^2} \frac{1}{c^2} \frac{\partial q'}{\partial t} \hat{q}
\]

where \( \lambda = -i\omega \) is the complex eigenvalue of the system:

- Real part \( \rightarrow \) Eigenfrequency;
- Imaginary part \( \rightarrow \) Growth Rate:
  - Growth Rate Positive \( \rightarrow \) Unstable Mode;
  - Growth Rate Negative \( \rightarrow \) Stable Mode.

Numerical method used \( \rightarrow \) Arnoldi ARPACK
Simulation Code \( \rightarrow \) COMSOL Multiphysics
Application

\[ q'(t) \frac{t}{q} = -k \left[ \mu_4 \left( \frac{u_i(t - \tau)}{\bar{u}_i} \right)^5 + \mu_2 \left( \frac{u_i(t - \tau)}{\bar{u}_i} \right)^3 + \mu_0 \frac{u_i(t - \tau)}{\bar{u}_i} \right] \]

\[ f = 66.7 \text{ Hz} \]
Weakly non linear analysis: algorithm procedure

Steps to get the bifurcation diagram:

1. The interaction index $k$ is the control parameter;
2. Define the amplitude $r$ ($= |\hat{u}/\bar{u}|$) by guess or starting from zero;
3. For each value of $r$ the eigenvalue problem is solved and the complex eigenfrequency is detected;
4. Vary the amplitude $r$ until the growth rate is zero;
5. The corresponding amplitude $r$ identifies a limit cycle solution;
6. Change the control parameter and start again.

Simulation Code  -->  COMSOL Multiphysics
Physics --> “Pressure Acoustics” of Module “Acoustics”
Cannular Configuration: subcritical bifurcation

\[
\frac{q'}{q} = -k \left[ \mu_4 \left( \frac{u'(t-\tau)}{u} \right)^5 + \mu_2 \left( \frac{u'(t-\tau)}{u} \right)^3 + \mu_0 \frac{u'(t-\tau)}{u} \right]
\]

\[
\mu_4 = -1 \quad \mu_2 = 1 \quad \mu_0 = 0.2
\]
Temperature increases from 774 K to 2350 K across the flame
Closed-end inlet and outlet boundary conditions, $u' = 0$
Weakly non linear analysis: algorithm procedure

Steps to get the bifurcation diagram:

1. The interaction index $k$ is the control parameter;
2. Define the amplitude $r (\equiv |\bar{u}/\bar{u}|)$ by guess or starting from zero;
3. For each value of $r$ the eigenvalue problem is solved and the complex eigenfrequency is detected;
4. Vary the amplitude $r$ until the growth rate is zero;
5. The corresponding amplitude $r$ identifies a limit cycle solution;
6. Change the control parameter and start again.
Annular Configuration: First azimuthal mode

1st Azimuthal Mode (735 Hz)
Annular Configuration: subcritical bifurcation

\[
\frac{q'}{q} = -k \left[ \mu_4 \left( \frac{u'(t-\tau)}{u} \right)^5 + \mu_2 \left( \frac{u'(t-\tau)}{u} \right)^3 + \mu_0 \frac{u'(t-\tau)}{u} \right]
\]

\[\mu_4 = -1 \quad \mu_2 = 1 \quad \mu_0 = 0.2\]
Annular Configuration: supercritical bifurcation

\[
\frac{q'(t)}{\overline{q}} = -\kappa \left[ \mu_2 \left( \frac{u'(t - \tau)}{\overline{u}} \right)^3 + \mu_0 \frac{u'(t - \tau)}{\overline{u}} \right]
\]

\[\mu_2 = -1; \, \mu_0 = 0.2\]
Experimental FDF – Polynomial Fitting

Ref. Dowling, JFM 1999

\[ \frac{\hat{q}}{\bar{q}} = \frac{\hat{u}}{\bar{u}} a(r, \omega) F(\omega) \exp(-i\omega \tau) \]

Fourth order polynomial regression of the normalized gain \(a(r,\omega)\)

\[ NFTF_{fit} = 0.12r^4 - 0.48r^3 + 0.47r^2 - 0.16r + 1.00 \]
Conclusion

- A weakly non-linear analysis procedure has been implemented in an FEM 3D Helmholtz solver for a cannular configuration.

- Two different analytical non-linear flame transfer function have been used.

- Bifurcation diagrams of the first unstable mode have been computed using the flame interaction index $\kappa$ as control parameter.

- Depending on the FTF used, the system has manifested a subcritical and a supercritical bifurcation.
Future Work

- Introduction of a more realistic flame transfer function or flame describing function
- Investigation on the influence of spatial non homogeneities of flame
- Comparison with experimental data on practical machine
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Mathematical Model: NonLinear Flame Model

\[ T^L_{\text{flame}}(\omega) = \frac{\hat{q}}{\bar{u}_i/\bar{u}} = -ke^{-i\omega \tau} \]

\[ \hat{q}^L = T^L_{\text{flame}}(\omega) \frac{\hat{u}_i}{\bar{u}_i} \]

\[ T^{NL}_{\text{flame}}(\omega, r) = T^L_{\text{flame}}(\omega) \cdot \text{NFTF}(r) \]

\[ \hat{q}^{NL} = \hat{q}^L \cdot \text{NFTF}(r) = T^L_{\text{flame}}(\omega) \frac{\hat{u}_i}{\bar{u}_i} \bar{q} \cdot \text{NFTF}(r) \]

\[ q'(t) = -k \left[ \mu_4 \left( \frac{u'_i(t-\tau)}{\bar{u}_i} \right)^5 + \mu_2 \left( \frac{u'_i(t-\tau)}{\bar{u}_i} \right)^3 + \mu_0 \frac{u'_i(t-\tau)}{\bar{u}_i} \right] \]

L --> linear
NL --> nonlinear
r --> amplitude
NFTF --> Nonlinear FTF

\[ \text{NFTF} = \frac{5}{8} \mu_4 r^4 + \frac{3}{4} \mu_2 r^2 + \mu_0 \]