The Spherical Design Algorithm in the Numerical Simulation of Fibre-Reinforced Biological Tissues

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Articular Cartilage

- Living Cells (proteoglycans, chondrocytes) constituting the ECM.
- Fibres of Collagen statistically distributed
- Fluid (mainly water) passing through and escaping from the tissue
\[ \hat{\Psi}(\xi, \Theta) = \frac{1}{Z(\xi)} \exp \left( -\frac{[\Theta - Q(\xi)]^2}{2[\sigma(\xi)]^2} \right) \]

\[ S_2^2 B := \{ M \in T_x B : \| M \| = 1 \} \]

\[ M = \sin(\Theta) \cos(\Phi) E_1 + \sin(\Theta) \sin(\Phi) E_2 + \cos(\Theta) E_3 \]
Mechanical Behaviour - Unconfined Compression

\[
\begin{align*}
\dot{J} + \text{Div} \left( -\hat{K}(F) \text{Grad} p \right) &= 0 \\
\text{Div} \left( -J \rho g^{-1} F^{-T} + \hat{P}_{sc}(F) \right) &= 0
\end{align*}
\]

\[
\hat{K}(F) = K_i(C, \xi) + \alpha(C, \xi)\hat{Z}(C)
\]

\[
Z = \hat{Z}(C) = \int_{S^2_B} \Psi(M) \frac{M \otimes M}{I_4(C, M)}.
\]

\[
\hat{P}_{sc}(F) = P_i(F) + FS_a
\]

\[
S_a = 2\phi_{1sR} \int_{S^2_B} \Psi(M) \mathcal{H}(I_4 - 1)[I_4 - 1]A.
\]

in which \( A = M \otimes M, I_4 = C : A. \)
The Spherical Design Algorithm

Let $f$ be any (scalar, vector, or tensor) function defined over $\mathbb{S}^2 \mathcal{B}$. Then $f(X, M) = \hat{f}(X, \Theta, \Phi)$ with $(\Theta, \Phi) \in \mathcal{D} = [0, \pi] \times [0, 2\pi]$.

\[
\int_{\mathbb{S}^2 \mathcal{B}} f(M) = \int \int_{\mathcal{D}} \hat{f}(\Theta, \Phi) \sin(\Theta) d\Theta d\Phi
\]

\[
\int \int_{\mathcal{D}} \hat{f}(\Theta, \Phi) \sin(\Theta) d\Theta d\Phi \simeq \frac{4\pi}{N} \sum_{i=1}^{m} \sum_{j=1}^{n} \hat{f}(X_{ij})
\]

with $X_{ij} = (\Theta_i, \Phi_j) \in \mathcal{D}$, with $i = 1, \ldots, m$ and $j = 1, \ldots, n$, $N = mn$. 
The Proper Choice of the Integration Points

\[
(Z_0)^{11} = (Z_0)^{22} = \pi \int_0^\pi \hat{\Psi}(\xi, \Theta)[\sin(\Theta)]^3 d\Theta,
\]

\[
(Z_0)^{12} = (Z_0)^{13} = (Z_0)^{23} = 0,
\]

\[
(Z_0)^{33} = 1 - 2(Z_0)^{11}.
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Benchmark Tests

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\end{align*}
\]

\[
\begin{align*}
\chi^3 &= H + w, & Q.E_3 &= 0, & \forall X \in \Gamma_u \\
-p &= 0, & P.N &= 0, & \forall X \in \Gamma_w \\
\chi^3 &= 0, & Q.(-E_3) &= 0, & \forall X \in \Gamma_1
\end{align*}
\]

\[w(t) = -0.2L \frac{t}{T}.
\]
## Benchmark Tests

<table>
<thead>
<tr>
<th>Name</th>
<th>Performed integration</th>
<th>$N$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sim-1</td>
<td>SDA</td>
<td>50 equidistributed</td>
</tr>
<tr>
<td>Sim-2</td>
<td>SDA</td>
<td>200 equidistributed</td>
</tr>
<tr>
<td>Sim-3</td>
<td>Matlab Integration, external call to Matlab</td>
<td>//</td>
</tr>
<tr>
<td>Sim-4</td>
<td>SDA</td>
<td>21 Sloane points</td>
</tr>
<tr>
<td>Sim-5</td>
<td>SDA</td>
<td>120 Sloane points</td>
</tr>
<tr>
<td>Sim-6</td>
<td>SDA</td>
<td>41, $(\Theta, \Phi) \in I \times J$</td>
</tr>
</tbody>
</table>

\[
\Theta \in I = \left\{ 0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{2\pi}{5}, \frac{\pi}{2} \right\},
\]

\[
\Phi \in J = \left\{ 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi, \frac{5\pi}{4}, \frac{3\pi}{2}, \frac{7\pi}{4} \right\}.
\]
Results - Radial Permeability

![Graphs showing radial permeability results for different simulations and a comparison with Matlab.](image)

<table>
<thead>
<tr>
<th></th>
<th>Sim-1</th>
<th>Sim-2</th>
<th>Sim-4</th>
<th>Sim-5</th>
<th>Sim-6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Comp. Time</td>
<td>5 min 15 s</td>
<td>≈ 40 s</td>
<td>55 s</td>
<td>2 min 20 s</td>
<td>≈ 30 s</td>
</tr>
<tr>
<td>Memory [Gb]</td>
<td>2.52</td>
<td>2.54</td>
<td>2.53</td>
<td>1.44</td>
<td>1.21</td>
</tr>
</tbody>
</table>

Sim-3 Computational time: 3 h 40 min
Sim-3 Memory: 2.4 Gb
Results - Axial Permeability

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<th>Sim-5</th>
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</tr>
</thead>
<tbody>
<tr>
<td>Comp. Time</td>
<td>5 min 15 s</td>
<td>≈ 40 s</td>
<td>55 s</td>
<td>1 min 20 s</td>
<td>18 s</td>
</tr>
<tr>
<td>Memory [Gb]</td>
<td>2.52</td>
<td>2.54</td>
<td>2.53</td>
<td>1.44</td>
<td>1.21</td>
</tr>
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</table>
Figure: Radial (left) and Axial (right) stress components related to fibres
Conclusions

- If the quadrature points are properly chosen, an internal implementation of the SDA is in general preferable to an external Matlab call.
- To validate the point set $\mathcal{I} \times \mathcal{J}$ for a more general computational and mathematical setting, we need to test it on a wider range of benchmark problems and constitutive laws.

Some Citations

Thank You
For Your Kind
Attention