Stationary Solutions to the Heat Equation

We consider heat transport, including non-linear source terms with Dirichlet boundary conditions and the physical parameters set to unity, we study the following Time Dependent (TD) dimensionless, non-linear equation in terms of stationary solutions.

The linear ($\varepsilon = 0$) problem have a simple analytic stationary solution. We then ($\varepsilon \neq 0$) compare with the built-in stationary solvers in Comsol [2]. Applying the DFPM we have

Convergence to the Stationary Solutions

The Dynamical Functional Particle Method (DFPM)

Consider the abstract equation for an unknown $v$, $\min_v E(v)$.

$$ F(v) = -\delta E(v)/\delta v = 0. $$

We introduce a parameter $\tau$ and define a new time dependent differential equation, such that $u \rightarrow v$ when $\tau \rightarrow \infty$,

$$ u_{\tau\tau} + \eta u_\tau = F(u). $$

We recognize a second-order damped system where $\eta$ represents the damping. However, the method is suitable for problems far from this mechanical analogue [1].

Motivation

It is generally challenging to find solutions to non-linear equations. Comsol built-in stationary solvers sometimes fails and also require good initial guesses. Yet, this situation occurs in multiple applications in science and technology. Adding constraints on the solution implies further difficulties for numerical methods.

Constraints

The DFPM can be used for additional constraints, $G_j(u) = 0$,

$$ u_{\tau\tau} + \eta u_\tau = F(u) + \sum_j \lambda_j G_j(u), $$

where $\lambda_j$ are Lagrange multipliers and

$$ G_j(u) = \delta G_j(u)/\delta u, $$

are defined by the functional derivative of a constraint functional [3].

Non-linear Schrödinger Equation (NLSE)

In modern physics, the stationary NLSE may have the form

$$ -\frac{\hbar^2}{2M} \nabla^2 \varphi + U_0 |\varphi|^2 \varphi = \mu \varphi, $$

for the complex wavefunction $\varphi$. Which is constrained by the normalization condition

$$ \int |\varphi|^2 dV = N_\varphi, $$

where $N_\varphi$ is the number of atoms, $M$ is the mass of an atom, and the atom-atom mean field interaction parameter $U_0$ can be varied in sign and amplitude. We recently solved this problem with DFPM for a ring geometry [3].

Future Work

In an ongoing project we use Comsol to investigate more realistic geometries [4].

Example of a dark soliton in a torus-shaped 3D Bose-Einstein condensate (BEC).

Can we implement DFPM efficiently in Comsol to find stationary solutions with constraints?

References