Space-time Formulation for Finite-Element Modeling of Superconductors

F. Grilli[*,1], F. Sirois[1], M. Laforest[1], and S. P. Ashworth[2]
[1]Ecole Polytechnique de Montréal, [2]Los Alamos National Laboratory
*Corresponding author: Ecole Polytechnique de Montréal, P.O. BOX 6079 Station Centre-ville, Montréal (QC), Canada H3C 3A7, f.grilli@polymtl.ca

Abstract: In this paper we present a new model for computing the current density and field distributions in superconductors by means of a periodic space-time formulation for finite elements (FE). By considering a space dimension as time, we can use a static model to solve a time dependent problem. This allows overcoming one of the major problems of FE modeling of superconductors: the length of simulations, even for relatively simple cases. We present our first results and compare them to those obtained with a 'standard' time-dependent method and with analytical solutions.
Keywords: High-temperature superconductors, ac losses, space-time.

1. Introduction

Finite-element calculations have proved to be capable of accurately predicting the ac losses in high-temperature superconductor (HTS) devices of increasing complexity. Different formulations (using different state variables) are possible, but in general the FE models solve Faraday's law (which contains partial derivatives with respect to position and time), using a non-linear resistivity for the superconductor material:

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{1}$$

$$\rho(J) = \frac{E_c}{J_c}\left|\frac{J}{J_c}\right|^{n-1} \tag{2}$$

The presence of a non-linear resistivity does not allow for a steady-state time harmonic solution, so the standard approach is to solve it as a time-dependent problem. This means that several time-steps have to be solved: typically two cycles are solved to eliminate the effects of the transient from the initial condition (generally a material in its virgin state). This approach results in a lengthy solving process. Even simple problems like a rectangular wire carrying transport current and/or subjected to an external field require several minutes of computation on fast workstations. Real superconductors as the recently developed YBCO coated conductors have a quite complex structure and may involve the presence of other complex materials, for example substrates with magnetic properties. Superconducting devices like coils and cables consist of a large number of windings and tapes and even if the geometry can be simplified by an appropriate use of symmetry and periodicity conditions, the number of nodes is still quite large. In addition to that, it has to be kept in mind that FE tools can be used for optimizing the design of a given device. This means that several configurations using different geometrical and physical parameters have to be tested. It is clear that with the present approach the computation times are unreasonable in most cases. Therefore the search for an alternative, faster, and equally reliable way of computing current and field profiles as well as ac losses in superconducting devices is mandatory.

In this framework, we started developing, implementing and testing a new model based on a periodic space-time (PST) formulation, where one of the dimensions of the simulated domain represents the time [1,2]. This allows one to solve the problem as a static one, which can be better handled by the solvers and is likely to be faster.

In this paper we present the first results of this project, showing that the problem is well posed and that there is a potential for obtaining a faster simulation tool. Three different cases are considered: (i) an infinite slab in applied ac magnetic field; (ii) a round conductor carrying ac transport current; (iii) three stacked conductors, each carrying a different transport current.

2. Infinite slab in ac magnetic field

As a first test, we considered the well-known problem of an infinite superconducting slab (of width *2a*) subjected to an external ac field parallel to the slab's face [1]. In this case Faraday's equation reduces to a scalar diffusion equation

$$-\frac{\partial}{\partial x}\left[\frac{\rho}{\mu_0}\frac{\partial B}{\partial x}\right]+\frac{\partial B}{\partial t}=0 \qquad (3)$$

where the diffusion coefficient ρ/μ_0 is highly non-linear. Figure 1 schematically shows the applied boundary conditions in the PST model, where the y axis represents the time.

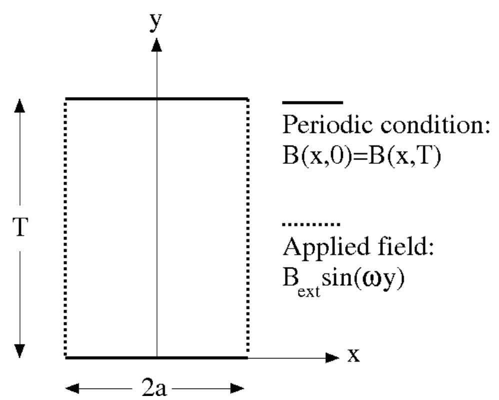

Figure 1. Boundary conditions for the PST model of a superconducting slab in external field.

At the slab's edge the (time-dependent) magnetic field is equal to the applied one, so on the vertical boundaries of the geometry the Dirichlet boundary condition $B=B_0\sin(\omega y)$ is applied. Since we are considering a periodic external magnetic field (in this case sinusoidal), we have to impose that the field at $y=0$ is equal to the field at $y=T$, where T represents the period for the considered frequency. The problem can be further reduced in size by considering only one half-period and imposing anti-periodicity conditions on the two horizontal lines. In addition, we want to keep the simulated geometry close to square, in order to have well-shaped mesh elements (in general the space and time scales may differ by several orders of magnitude): this can easily be done by scaling the time by a constant factor ($y=ct$), which is inserted directly in the equations appearing in the Physics Settings.

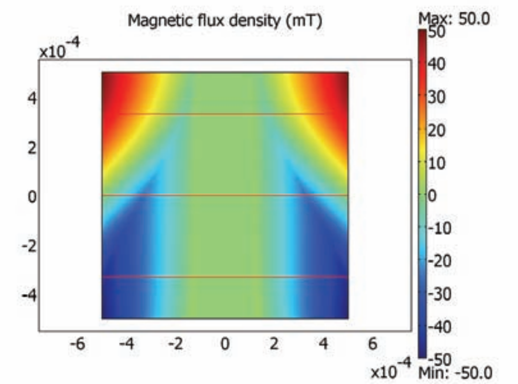

Figure 2. Distribution of magnetic flux density in the slab for an applied field of 50 mT at 500 Hz.

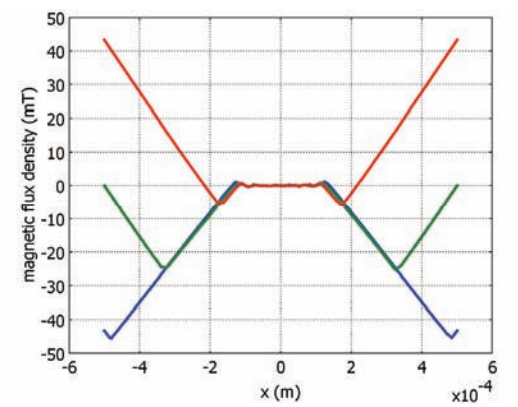

Figure 3. Profiles of the magnetic flux along the slab's width at three different time instants, corresponding to the three horizontal lines in Figure 2**.**

A typical distribution of the magnetic flux density is shown in Figure 2, whereas Figure 3 shows the field profiles along the slab's width for three particular instants of the ac cycle, represented by the horizontal lines in Figure 2. These results are typical for this problem and agree well with those computed with standard transient 2D models [4] and with analytical predictions [1] (not shown here).

We used the PST model to compute the ac losses. Generally the losses are computed by integrating the power dissipation over the superconductor's section and integrating over a cycle. In this case, since the time is represented by the y axis, it is sufficient to integrate the power dissipation on the simulated geometry. The loss values are in excellent agreement with those computed with 2D transient method as well as with analytical formulae – see Figure 4a.

We also compared the computation time. In the case of the PST model, it is possible to simulate a case starting from a solution computed previously for a slightly different value of a given parameter. For example, if we are varying the magnetic field amplitude, the solution for a field of 20 mT can be computed from that for a field of 10 mT. Since the solution does not change much, the computing time is expected to be lower than the time for a solution computed from the virgin state (when all the variables are initialized to zero). This is what we observed, as shown in Figure 4b.

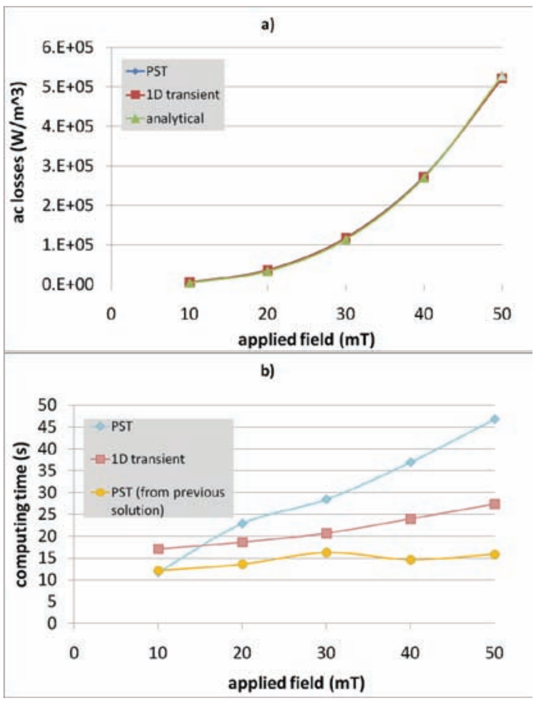

Figure 4. Comparison of ac loss values (a) and computing time (b) between the PST and the standard 1D transient model for the case of an infinite slab subjected to an ac parallel field. For the PST model computing times are shown for solutions computed starting from virgin state and from the solution corresponding to the previous applied field value.

This reduction of computation time can be very useful in view of using this approach for device optimization, where typically several parametric studies have to be performed. It has to be kept in mind that the slab problem is quite simple and the gain in computation time might not seem important in absolute terms. However, it is expected to be more important for more complex problems: the results obtained for the round conductor (see next case) seem to go in the right direction. Computation times refer to a workstation equipped with a 2.4 GHZ processor and 3 GB of RAM.

3. Round conductor carrying ac current

As more complex case we considered a round superconductor carrying ac current. This is an example of a 2D transient problem (the conductor's cross section) that becomes a 3D static one. The implementation is similar to that of the slab, but this time the boundary conditions for the two magnetic field components are used to impose the desired transport current I_{ext}, according to Biot-Savart's law: $H_{\Phi}=I_{ext}/2\pi r$, where r is the cylinder's radius.

An example of the current density distribution is shown in Figure 5, where each "slice" represents a particular time instant (the cylinder's axis representing the time direction). As in the case of the slab, we obtained good agreement with the losses computed by the corresponding 2D transient model and the analytical formulae developed by Norris [5] – see Figure 6. (In Norris's formulation, losses diverge for $I=I_c$, this is why at high currents the losses are higher than the simulated ones.) The gain in computation time by starting from a previously computed solution is even more important than in the case of the slab, confirming the potential of the PST approach to be a fast, reliable tool for ac loss computation in HTS devices – see Figure 6b.

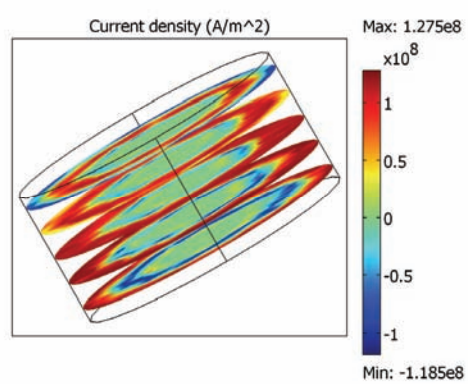

Figure 5. Current density distribution in a round conductor at different instants of the cycle. The

amplitude of the transport current is 200 A, the critical current is 314 A.

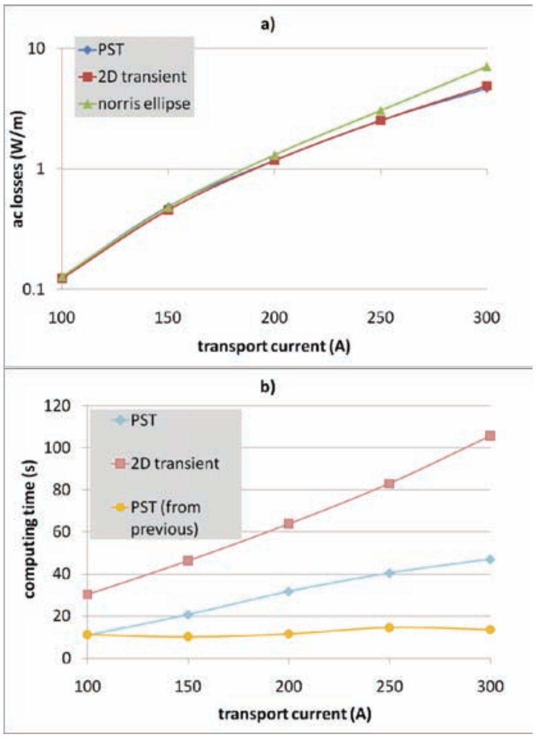

Figure 6. Comparison of ac loss values (top) and computing time (bottom) between the PST and the standard 2D transient model for the case of a round conductor carrying ac current. For the PST model computing times are shown for solutions computed from virgin state and from the solution corresponding to the previous current value.

4. Multiple conductors carrying ac current

The cases presented in the previous sections have a common characteristic: the value of the state variables (magnetic field) on the conductor's boundary is known and it is used to impose the desired field or current. In the general case of multiple conductors carrying arbitrary currents this is no longer true and the desired current has to be imposed by integral constraints. In this section we describe the different approaches we have followed to do that and we show the results for the approach with which we obtained reasonable results.

4.1 Different approaches

The imposition of current constraints in COMSOL is not straightforward and for this purpose we have tried three different approaches, schematically represented in Figure 7 (for sake of clarity, the 1-D case is represented):

1. Use a finite difference approximation for the $\partial B/\partial t$ term, which implies no mesh at all along the z axis (corresponding to the time). Only a conventional n-D mesh is required here, and it will be repeated once for each time step of the periodic solution. On each plane $z=z_n$ we have to impose that the current is equal to the value at that "time".
2. Mesh the whole space-time domain by extrusion (i.e. one layer per time steps), and approximate $\partial B/\partial t$ with a weak formulation. With a proper choice of base functions on elements, current constraints only need to be imposed at the $z=z_n$ planes, as above.
3. Use an arbitrary mesh over the space-time domain, and approximate $\partial B/\partial t$ with a weak formulation. Current constraint imposition becomes more delicate, because we need to sweep the z axis in order to catch the contribution of all elements (no "floating elements" allowed, otherwise the matrix will be singular). In 2D, this means integrating the current density $J(x,y,z)$ (which is directly available from the two magnetic field components $H_x(x,y,z)$ and $H_y(x,y,z)$) along x and y to create a function $I(z)$ and impose that this function is equal to the desired current $I_{ext}sin(\omega z)$.

So far, only the first option could be properly implemented in COMSOL. Even if it is far from being optimal at this stage (for a series of reasons outlined below), it proved the correctness of the concept of space-time formulation in the case of multiple conductors.

The second and third methods were also implemented in COMSOL, but even after many iterations with the technical support, a number of problems still prevent us to obtain correct solutions (although it does converge to some solution). In particular, with the second method, the current constraint seems to never to be correctly satisfied internally by COMSOL, although it is in principle set correctly. Because of that, we did not spend too much effort on it.

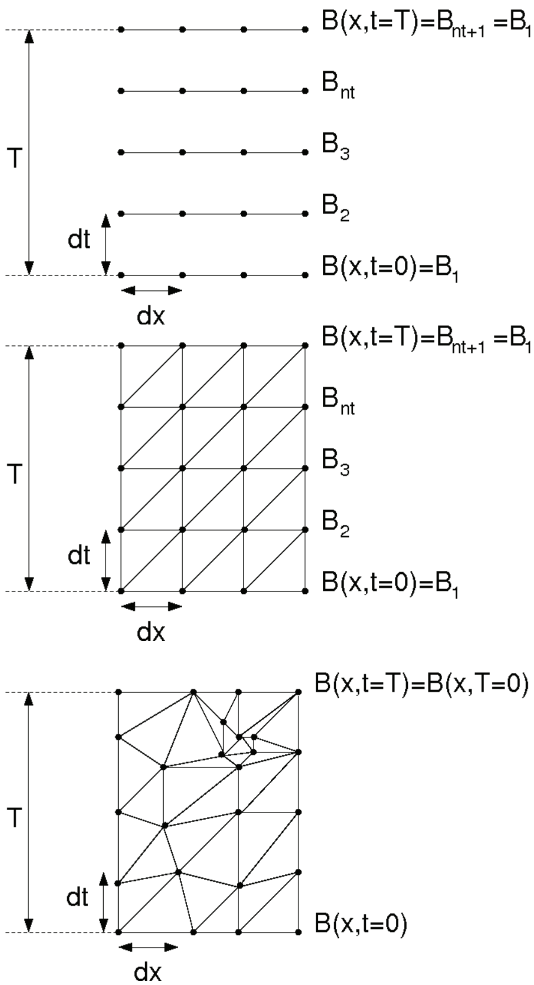

Figure 7. Schematic view of the mesh used in the three different space-time approaches. For sake of clarity, the 1-D case is shown. Top: with the method of the finite differences, no explicit solutions exist between layers *j* and *j+1*. Center: with method 2, the solution is well defined between layers in every point *(x,t)*. Bottom: with method 3, the domain is freely meshed; adaptive mesh algorithm can be used to refine the mesh in the most critical regions.

Most of our efforts were rather spent on the third method, which has the best potential in terms of flexibility. However, this flexibility comes at the expense of increased complexity for imposing the current constraint. But once again, even if in COMSOL there are all tools to perform the *J* integration and then obtain *I(z)* (through the features *projection / extrusion coupling variables*) and impose the constraint $I(z)=I_{ext}\sin(\omega z)$ using a COMSOL built-in 1D weak form physics, the results obtained are physically incorrect. Unfortunately, the use of these features is little documented in COMSOL's manual, and even after a deep investigation with the help of COMSOL's support, it does not appear to have a reliable way to solve the problems encountered. Note that some of these problems are documented COMSOL bugs (e.g. some cross-meshing operations when using a projection coupling variable from a 3-D domain to a 2-D domain in a different geometry), and should be resolved in a future release of the software.

4.1 Results with finite difference approach

In this section we describe the details of the finite difference approach ('method 1') and show results for a case of three stacked conductors carrying different currents.

We start from the general diffusion equation:

$$\nabla \times \frac{\rho}{\mu_0} \nabla \times B = -\frac{\partial B}{\partial t} \tag{4}$$

with the Neumann boundary conditions

$$\frac{\rho}{\mu_0}\frac{\partial B}{\partial n} = 0 \tag{5}$$

where *n* is the unit vector normal to the external face of the domain. We approximate the variation rate of the magnetic flux with a finite difference:

$$\frac{\partial B}{\partial t} \approx \frac{B(t+\Delta t) - B(t-\Delta t)}{2\Delta t} \tag{6}$$

For each layer *j*, we have a solution $B_j = B(x,y,t_j)$, corresponding to an equation of the form:

$$\nabla \times \frac{\rho}{\mu_0} \nabla \times B_j = \frac{B_{j+1} - B_{j-1}}{2\Delta t} \tag{7}$$

Using anti-symmetric boundary conditions with n_t points per half-cycle, this allows settingup a problem involving n_t "layers" coupled together by the B-terms appearing on the right hand-side, i.e.

$$\nabla \times \frac{\rho}{\mu_0} \nabla \times B_1 = \frac{B_2 - B_0}{2\Delta t}$$
$$\nabla \times \frac{\rho}{\mu_0} \nabla \times B_2 = \frac{B_3 - B_1}{2\Delta t} \qquad (8)$$
$$\cdots$$
$$\nabla \times \frac{\rho}{\mu_0} \nabla \times B_{n_t} = \frac{B_{n_t+1} - B_{n_t-1}}{2\Delta t}$$

with $B_0 = -B_{n_t}$ and $B_{n_t+1} = -B_1$ because of anti-symmetry.

In COMSOL, this approach can be implemented by using extrusion coupling variables to make the variables B_{j+1}, B_{j-1} available in the j^{th} "layer" (or "geometry" in COMSOL's terminology) and by inserting the finite difference expressions as source terms in a COMSOL's magnetostatic application mode. This means that instead of having a full 3D geometry, we have a set of n_t coupled 2D problems to solve.

The system of coupled equations can be visually represented by the matrix of Figure 8, where each block A_j represents the full equation system of an isolated layer j (which is the only matrix solved for in the time transient formulation). Therefore, we clearly see that in the problem, the resulting matrix is much larger than the one used in conventional time transient, but this is compensated by the fact that it only has to be solved once instead of many hundred times (i.e. for every time steps of the transient solution). In addition, the 'space-time matrix' is not full, but the diagonal is much wider than that of the single A matrix. Finally, we observe that there are also elements in the corners of the 'space-time matrix', which is typical of a periodic structure.

In COMSOL this approach can be easily implemented by using extrusion coupling variables to make the variables B_{j+1}, B_{j-1} available in the j^{th} layer (i.e. the j^{th} geometry in terms of COMSOL's language) and by inserting the finite difference expressions as source terms (F) in a COMSOL's magnetostatic application mode.

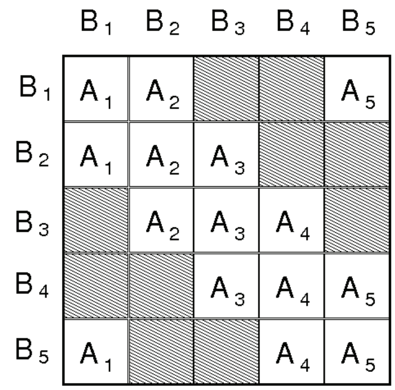

Figure 8. Final system matrix for the finite difference space-time approach, in the particular case of five time intervals in a half-cycle. The shaded parts represent the zeros of the matrix.

We have started comparing the results of this finite difference implementation with those obtained with the standard 2-D transient model. As an example, we considered three stacked square conductors, each carrying different currents (from top to bottom: 80 A, 70 A, and 60 A, respectively, currents in phase). Figure 9 shows the current density distribution in the three conductors at the peak of the current. We purposely display the full COMSOL graphical interface to show that the problem actually consists of several 2D geometries (in this case 16), each corresponding to a different time instant in the half-cycle.

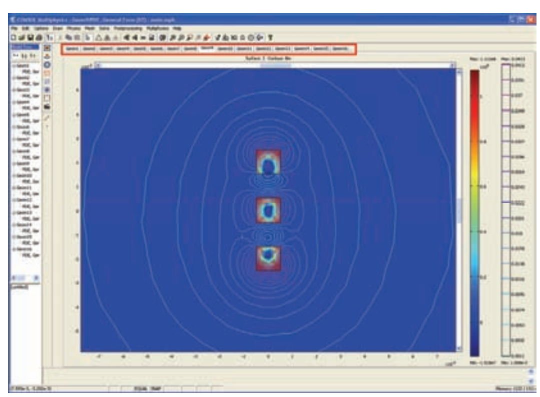

Figure 9. Example of COMSOL graphical interface for the problem of multiple conductors with the finite difference method. It can be noticed that the problem consists of several (in this case 16) coupled 2D problems – see red rectangle showing the different geometries.

Figure 10 shows the comparison of the time evolution of the losses computed with the finite difference model and with the standard 2D transient model: the agreement is very good. The computation time, however, is much larger than for the standard transient model. This is likely to be caused by the strong form of the finite time difference time derivatives, which are known to be difficult to satisfy in problems with very nonlinear properties, such as here. This is why a fully weak formulation would be highly desirable, such as methods 2 and 3 described above.

In conclusion, the finite difference approach, while showing the correctness of the principle of using the PST model for simulating multiple conductors, is not a viable solution to solve practical problems.

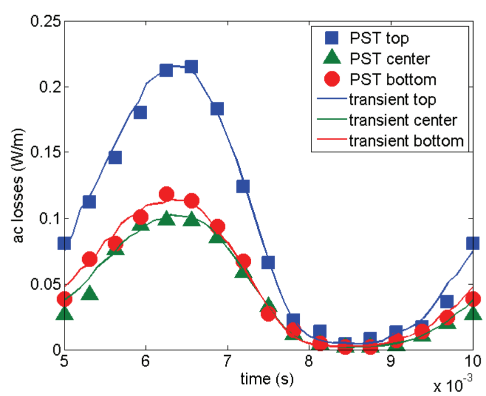

Figure 10. Instantaneous AC losses (over a half-cycle, frequency 100 Hz) for three stacked superconductors carrying current of 80, 70 and 60 A computed with the PST formulation (symbols) and the standard transient model (lines).

5. Conclusions

The tests illustrated in this report showed the correctness of the periodic space-time approach to solve transient problems for high-temperature superconductors. The implemented model, while showing results in good agreement with standard transient models, has still to be refined and optimized at two levels:

1. The 'method 3' is by far the most flexible and interesting for applications. Additional work will be done to try to implement the space-time model using this approach. This will involve a deeper knowledge of the COMSOL software and a more intense collaboration with the support people. We hope that will be possible now that the formulation has been validated with 'method 1'.
2. After several years of use, the parameters of the transient model are now optimized to handle problem with superconductors [4, 5]. This has still to be done for the space-time model. In particular, optimizing the settings of the solvers and the mesh will allow having much faster simulations.

6. References

1. T. Nakata, N. Takahashi, K. Fujiwara, A. Ahagon, *IEEE Transactions on Magnetics*, **Vol. 25**, no. 5, pp. 4150-4152 (1989).
2. T. Nakata, N. Takahashi, K. Muramatsu, H. Ohashi, H. L. Zhu, *IEEE Transactions on Magnetics*, **Vol. 31**, no. 3, pp. 1416-1419, (1995).
3. M. N. Wilson, *Superconducting Magnets*, pp. 162-164. Clarendon Press, Oxford UK (1983).
4. R. Brambilla, F. Grilli, L. Martini, *Superconductor Science and Technology*, **Vol. 20**, pp. 16-24 (2007).
5. F. Sirois, F. Grilli, *IEEE Transactions on Applied Superconductivity*, Sept. 2008, at press.
6. W. T. Norris, *Journal of Physics D: Applied Physics*, **Vol. 3**, pp. 489-507 (1970).

7. Acknowledgements

This work was supported by the Mathematics of Information Technology and Complex System (MITACS) network and by the U.S. DOE Office of Electricity Delivery and Energy Reliability.