Solving Distributed Optimal Control Problems of the Unsteady Burgers Equation in COMSOL Multiphysics

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1 Problem Setting

- Burgers equation as a first approximation to complex diffusion convection phenomena and as simplified model for turbulence and in shock waves.
- Analysis and numerical approximation of optimal control problems for Burgers equation are important for the development of numerical methods for optimal control of more complicated models in fluid dynamics like Navier-Stokes equations.
- In contrast to linear parabolic control problems, the optimal control problem for the Burgers equation is a non-convex problem with multiple local minima due to nonlinearity of the differential equation. Numerical methods can only compute minima close to the starting points.

2 Optimal control of Burgers equations without inequality constraints

The distributed control problem without inequality constraints [7]:

\[
\min \ J(y, u) = \frac{1}{2} \| y - y_f \|^2_{L^2} + \frac{\alpha}{2} \| u \|^2_{L^2}
\]

s.t. \ y_0 + y_p - \nu y - f + b u = 0 \text{ in } Q,
\[ y = g \text{ on } \Sigma, \]
\[ y = 0 \text{ on } \Omega, \]
\[ p = 0 \text{ on } \Omega, \]
\[ y = y_0 \text{ in } \Omega \times [0, T]. \]

\[ y : \text{ state, } \quad y_p : \text{ the optimal control, } \quad \nu \text{ : the gradient method; } \quad \text{the state equation was solved for forwards and the adjoint equation backward in time and solving the the whole optimality system as a semismooth Newton equation produces satisfactory results for the Burgers equation.} \]

2.1 One-shot approach: treating the reverse time direction by simultaneous space-time discretization

- In the sequential approach optimality system is solved iteratively using the gradient method; the state equation was solved for forwards and the adjoint equation backwards for \( y \) until convergence.
- In one-shot approach, the optimality system in the whole space-time cylinder is solved as an elliptic (biharmonic) equation by interpreting the time as an additional space variable [2].

\[
y_0 + y_p - \nu y - f + b u = 0 \text{ in } Q,
\]
\[ p = 0 \text{ on } \Omega, \]
\[ y = y_0 \text{ in } \Omega \times [0, T]. \]

- Adaptive elliptic solver with adaptation.
- Nonadaptive elliptic solver femnan.

Numerical example [6]: \( \alpha = 0.01, \nu = 0.001, f = 0 \), the desired state \( y_f(0, x) = y_0 \) and the initial condition \( y_0 \) is located on the support set \((0, T) \times \left( \frac{1}{2}, \frac{3}{4} \right)\).

3 Optimal control of Burgers equation with inequality control constraints

Distributed optimal control problem with bilateral control constraints:

\[
\min \ J(y, u) = \frac{1}{2} \| y - y_f \|^2_{L^2} + \frac{\alpha}{2} \| u \|^2_{L^2}
\]

s.t. \ y_0 + y_p - \nu y - f + b u = 0 \text{ in } Q,
\[ y = g \text{ on } \Sigma, \]
\[ y = 0 \text{ on } \Omega, \]
\[ p = 0 \text{ on } \Omega, \]
\[ y = y_0 \text{ in } \Omega \times [0, T]. \]

\[ \alpha \geq 0, \quad \nu \geq 0, \quad f \geq 0, \quad b \geq 0. \]

4 Conclusions and extensions

- We have shown that the finite element package of COMSOL Multiphysics can be used for solving time-dependent non-linear optimal control problems.
- Both classical gradient based approach solving the state equation forward in time and the adjoint equation backward in time and solving the the whole optimality system as a semismooth Newton equation produces satisfactory results for the Burgers equation.
- Considering Burgers equation with state constraints as it was done in [3] for parabolic control problems.
- Application of the various stabilization techniques available in COMSOL Multiphysics to the Burgers equation.

References