Stefan’s Problem: Validation of a One-Dimensional Solid-Liquid Phase Change Heat Transfer Process

Wilson Ogo and Dominic Groulx
Department of Mechanical Engineering
Dalhousie University, Halifax, NS, Canada
Sensible Heat Storage:
A heat storage system that uses a heat storage medium, and where the addition or removal of heat results in a change in temperature.

Thermochemical Storage:
Storage of energy is the result of a chemical reaction.

Latent Heat Storage:
The storage of energy is the result of the phase change (solid-liquid or solid-solid) of a phase change material (PCM). The process happening over a small temperature range.
Finite Elements can be used to help in the design of Latent Heat Energy Storage Systems (LHESS):
- Determination of the application-dependent size of the LHESS;
- Choice of geometry;
- Heat Transfer enhancement (fins for example);
- Etc ...

A proper validation of the phase change behavior of the Phase Change Material (PCM) inside the LHESS is necessary to ensure the proper numerical treatment, especially when it comes to accounting for the total amount of energy stored in the system.
Geometry Studied

1D Stefan’s Problem
Geometry

$x = 0$

$t = 0$

$T_w$

Liquid

Solid, $T_m$

Melting front

PCM

Semi infinite medium
Analytical Solution

1D Stefan’s Problem
Governing Equations

- Heat Conduction in the liquid phase:
  \[
  \left( \rho C_p \right)_l \frac{\partial T_l}{\partial t} = k_l \frac{\partial^2 T_l}{\partial x^2}
  \]

- Boundary Conditions:
  \[
  T(x = 0, t > 0) = T_w
  \]
  \[
  T(x > \delta(t), t > 0) = T_m
  \]
  with \( \delta(t) \) being the solid-liquid interface position.

- Energy balance at the melting interface:
  \[
  -\rho_l L \frac{d\delta(t)}{dt} = k_l \frac{\partial T_l(\delta, t)}{\partial x} = -\rho_l L u_m
  \]
Solving the previous equations results in:

\[
\frac{T_l(x, t) - T_w}{T_m - T_w} = \frac{\text{erf}[x / 2\sqrt{\alpha_l t}]}{\text{erf}(\beta)} = \frac{\text{erf}(\eta)}{\text{erf}(\beta)}
\]

with \( \eta = \frac{x}{2\sqrt{\alpha_l t}} \)

and \( \beta \) determined by solving the following equation:

\[
\beta e^{\beta^2} \text{erf}(\beta) = \frac{\text{Ste}}{\sqrt{\pi}}, \text{ Ste = Stefan number = } \frac{C_{p,l}(T_w - T_m)}{L}
\]
Analytical Solution

The following can also be obtained analytically:

- Melting front position: \( \delta(t) = 2\eta \sqrt{\alpha_l t} \)
- Melting front velocity: \( u_m(t) = \frac{d\delta(t)}{dt} = \eta \sqrt{\alpha_l} \sqrt{\frac{1}{t}} \)
- Heat transfer rate at the solid-liquid interface:

\[
q''[\delta(t)] = k_l \frac{\partial T_l}{\partial x} \bigg|_{x=\delta(t)} = -\rho_l L \frac{\eta \sqrt{\alpha_l}}{\sqrt{t}}
\]
Numerical Modeling

2D Stefan’s Problem
Geometry

\[ x = 0 \]
\[ t = 0 \]

Max simulated time: 16 hrs

Insulation

\[ T_w = 350 \text{ K} \]

\[ L = 0.28 \text{ m} \]

\[ H = 0.1 \text{ m} \]

\[ T_m = 313 \text{ K} \]
The phase change material used in the validation study is **Paraffin wax**

<table>
<thead>
<tr>
<th>Property</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Thermal Conductivity</td>
<td>0.21 W/m·K</td>
</tr>
<tr>
<td>Heat Capacity</td>
<td>2.4 kJ/kg·K</td>
</tr>
<tr>
<td>Density</td>
<td>750 kg/m$^3$</td>
</tr>
<tr>
<td>Enthalpy of Fusion</td>
<td>175 kJ/kg</td>
</tr>
<tr>
<td>Melting Temperature Range</td>
<td>313 K to 316 K</td>
</tr>
</tbody>
</table>
Modeling in COMSOL

- Problem type: Transient thermal fluid*

- Model used: Heat Transfer in a Solid
  Transient Analysis

  This model encompasses:
  - Heat transfer by conduction.
    In Stefan’s Problem, convection is neglected in the liquid PCM
  - Modified using the Effective Heat Capacity Method.

- Geometry is considered 2D

* The treatment of phase change renders the problem non-linear as well.
Modified $C_p$ Method

$$C_p = \begin{cases} 
C_{p,s} & T < 313 \text{ K} \\
C_{p,\text{eff}} & 313 \text{ K} < T < 316 \text{ K} \\
C_{p,l} & T > 316 \text{ K} 
\end{cases}$$

Where

$$C_{p,\text{m}} = \frac{L}{(\Delta T_m)} + \frac{(C_{p,s} + C_{p,l})}{2}$$

Numerically

$$C_p = (2.5 + 60.5 \times (313 < T) - 60.5 \times (T > 316))$$

$C_{p,\text{eff}}$ = Effective $C_p$

$C_{p,s}$ = Solid phase $C_p$

$C_{p,l}$ = Liquid phase $C_p$

$L$ = Latent heat of fusion

$\Delta T_m$ = Melting Temperature range

$C_{p,\text{eff}} = 60.5 \text{ kJ/kg}$

$C_{p,s} = 2.4 \text{ kJ/kg}$

$C_{p,l} = 2.4 \text{ kJ/kg}$

$L = 175 \text{ kJ/kg}$
Modified $C_p$

PCM Melting Temperature Range: 313K-316K
Validation

Stefan’s Problem
Element Size

Distance from the wall (m) vs. Temperature (K)

- 30 mm
- 10 mm
- 3 mm
- 0.6 mm

Legend:
- Blue line: 30 mm
- Cyan line: 10 mm
- Dashed line: 3 mm
- Dotted line: 0.6 mm
Numerical vs Analytical
Effect of Melting Temperature Range

![Graph showing the effect of melting temperature range on distance from the wall.](image)

- **Analytical**
- **Melting range(313K-314K)**
- **Melting range(313K-315K)**
- **Melting range(313K-316K)**

Decrease in melting temperature range
Melting Front Position

![Graph showing the comparison between numerical and analytical methods for melting front position over time. The x-axis represents time in hours, and the y-axis represents melting front position in meters. The graph includes two lines: one for numerical data (dashed blue line) and another for analytical data (solid red line). The data points are marked with blue triangles and red squares, respectively.](Image)
Conclusion

- The physical processes encountered during transient phase change heat transfer, coupled with conduction, in a PCM can be modeled numerically using COMSOL Multiphysics;

- The appearance and the behavior of the melting front can be simulated by modifying the specific heat of the PCM to account for the increased amount of energy, in the form of latent heat of fusion, needed to melt the PCM over its melting temperature range.

- The validation showed the effect incorporating a mushy region in the physical modeling of the PCM had on the temperature profile in the liquid PCM and the melting front behavior.