Hybrid FEM-BEM approach for open boundary magnetostatic problems

Two- and three-dimensional formulations
Motivation

Have ever experienced this?

You have some electromagnetic problem ...

... and the area of interest is somewhere there ...

... or you do not know how to choose the exterior domain in order to keep the additional error small?

This may be fine as long as we are only interested in the properties of the electromagnetic field. But usually these equations form only a small part of a much larger system.

Is it possible to eliminate the degrees of freedom in the auxiliary domain?
Maxwell’s equations of magnetostatics:
\[
\nabla \cdot B = 0 \quad \Rightarrow \quad B = -\mu_0 (M + H) \\
\nabla \times H = 0 \quad \Rightarrow \quad H = -\nabla \varphi \\
\rightarrow \quad \Delta \varphi = \nabla \cdot M \quad \forall \mathbf{r} \in \mathbb{R}^n \\
\frac{\partial \varphi}{\partial \mathbf{n}} = \mathbf{n} \cdot M \quad \forall \mathbf{r} \in \partial V_{\text{mag}}
\]

Let’s try the decomposition \( \varphi = \varphi_1 + \varphi_2 \) with
\[
\Delta \varphi_1 = \nabla \cdot M \quad \forall \mathbf{r} \in V_{\text{mag}} \\
\frac{\partial \varphi_1}{\partial \mathbf{n}} = \mathbf{n} \cdot M \quad \forall \mathbf{r} \in \partial V_{\text{mag}} \\
\varphi_1 = 0 \quad \forall \mathbf{r} \in \mathbb{R}^n \setminus V_{\text{mag}} \\

\Delta \varphi_2 = 0 \quad \forall \mathbf{r} \in \mathbb{R}^n \\
\varphi_2(\mathbf{r}) = \int_{\partial V_{\text{mag}}} \varphi_1(\mathbf{r}') \frac{\partial}{\partial \mathbf{n}} G(\mathbf{r}, \mathbf{r}') d\mathbf{r}' \\
+ \left( \frac{1}{4\pi} \Omega(\mathbf{r}) - 1 \right) \varphi_1 \quad \forall \mathbf{r} \in \mathbb{R}^n
\]

Greens function \( G \) depends on the system dimension \( n \):
\[
G(r, r') = \begin{cases} 
\frac{1}{2\pi} \ln(|r - r'|) & n = 2 \\
\frac{1}{4\pi} \frac{1}{|r - r'|} & n = 3
\end{cases}
\]
Open boundary approaches

Implementation into COMSOL Multiphysics

No longer need for auxiliary domain
Solution is exact

\[
\Delta \varphi_1 = \nabla \cdot M \quad \forall \mathbf{r} \in V_{\text{mag}} \\
\frac{\partial}{\partial \hat{n}} \varphi_1 = \hat{n} \cdot M \quad \forall \mathbf{r} \in \partial V_{\text{mag}} \\
\varphi_1 = 0 \quad \forall \mathbf{r} \in \mathbb{R}^n \setminus V_{\text{mag}}
\]

\[
\Delta \varphi_2 = 0 \quad \forall \mathbf{r} \in V^n_{\text{mag}}
\]

\[
\varphi_2(\mathbf{r}) = \int_{\partial V_{\text{mag}}} \varphi_1(\mathbf{r}') \frac{\partial}{\partial \hat{n}} G(\mathbf{r}, \mathbf{r}') \, d\mathbf{r}' \\
+ \left( \frac{1}{4\pi} \Omega(\mathbf{r}) - 1 \right) \varphi_1 \quad \forall \mathbf{r} \in V^n_{\text{mag}}
\]

% Integration coupling variables
expr{1} = -phi1/(4*pi)* ... 
  (sign(x)*abs(nx)*(dest(x)-x) ... 
  +sign(y)*abs(ny)*(dest(y)-y) ... 
  +sign(z)*abs(nz)*(dest(z)-z))*( ... 
  (dest(x)-x)^2+ ... 
  (dest(y)-y)^2+ ... 
  (dest(z)-z)^2)^(-3/2);

fem.weak{1} = test(phi1x)*(phi1x - Mx) ... 
  test(phi1y)*(phi1y - My) ... 
  test(phi1z)*(phi1z - Mz);

\( \phi_1 \)-mode
\( \phi_2 \)-mode

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expr{1} = -phil/(4*pi)* ... 
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  +sign(y)*abs(ny)*(dest(y)-y) ... 
  +sign(z)*abs(nz)*(dest(z)-z))*( ... 
  (dest(x)-x)^2+ ... 
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\( \phi_1 \)-mode
\( \phi_2 \)-mode

hybrid FEM-BEM approach
Benchmark model

Outer regions

Domain

Solution

Dirichlet data

(a) Arbitrary exterior domain with angular surface ...

(b) ... that does not affect the solution of the magnetic stray field ...

(c) ... due to an appropriate choice of Dirichlet boundary conditions.
Benchmark model

**Error analysis**

\[
\Delta H^1(\varphi_h) = \| \varphi - \varphi_h \|_{H^1(V_{\text{mag}})} = \left( \int_{V_{\text{mag}}} | \nabla \varphi - \nabla \varphi_h |^2 \, dr \right)^{-1/2}
\]

<table>
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<th>#Elem.</th>
<th>#Bnd. Elem.</th>
<th>(\Delta H^1(\varphi_h))</th>
<th>(t_{\text{sol}} [s])</th>
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</table>

**Sparsity plots**

43725²-matrix

9966²-matrix

FEM

hybrid FEM-BEM
Influence of boundary topology

Homogeneously magnetized cube

\[
\varphi_2(r) = \int_{\partial V_{mag}} \varphi_1(r') \frac{\partial}{\partial \hat{n}} G(r, r') \, dr' + (\frac{1}{4\pi} \Omega(r) - 1) \varphi_1
\]

\[
\Omega(r) = \begin{cases} 
4\pi & \text{inner point} \\
2\pi & \text{along smooth surface}
\end{cases}
\]

...%

Solid angle, Boundaries
fem.appl{3}.mode = FlPDEWBoundary;

... % use values along edges as % boundary values

% Solid angle, Edges
fem.appl{4}.mode = FlPDEWEdge;

... % use values at corners as % boundary values
High aspect geometries

\[ \varphi_1(r) \approx \varphi_1(x, y) \]

Integral decomposition

\[
\int_{\Gamma_\perp} \varphi_1 \frac{\partial G}{\partial \hat{n}} \, dr' = - \frac{a_z}{4\pi} \int_{\Gamma_\perp} \varphi_1 \frac{\varphi_1 \, dx' \, dy'}{(\Delta r_{xy}^2 + a_z^2 / 4)^{3/2}}
\]

\[
\int_{\Gamma_{||}} \varphi_1 \frac{\partial G}{\partial \hat{n}} \, dr' = - \frac{a_z}{4\pi} \int_{\Gamma_{||}} \varphi_1 \frac{\hat{n} \cdot \Delta r_{xy}}{|\Delta r_{xy}|^3} \, dr'
\]

\[ \Delta r_{xy} = (x - x')\hat{x} + (y - y')\hat{y} \]

FEM-BEM: 0.17 s

FEM: 1.72 s

Increased performance due to reduced dimensionality, thickness only enters as numerical parameter.
Conclusion

- Hybrid FEM-BEM approaches can be implemented into COMSOL Multiphysics
- So far, for full three-dimensional problems, no increase of performance can be reported due to decreased sparsity of the matrix
- Increased performance for geometries of high aspect ratios

Outlook

- Further optimization of solver parameters and settings
- Implementation of hybrid FEM-BEM methods in ferromagnetic systems