Planar Geometry Ferrofluid Flows in Spatially Uniform Sinusoidally Time-varying Magnetic Fields

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Ferrofluids

• Ferrofluids
  – Nanosized particles in carrier liquid (diameter~10nm)
  – Super-paramagnetic, single domain particles
  – Coated with a surfactant (~2nm) to prevent agglomeration

• Applications
  – Hermetic seals (hard drives)
  – Magnetic hyperthermia for cancer treatment

Motivation

• Prior ferrofluid problems solved in COMSOL are usually in spherical and cylindrical geometries
• Ferrofluid pumping in planar geometry subjected to perpendicular and tangential magnetic fields
  – Well posed problem with analytical solutions
• Traditionally solved using mathematical packages such as Mathematica
  – Can COMSOL replicate these results?
Planar Geometry Setup

\[ \mathbf{v} = v_z(x) \mathbf{i}_z, \quad \omega = \omega_y(x) \mathbf{i}_y \]
How to impose $B_x$ field?

(a) DC Current source gives $H = NI/s$

(b) $V = \Lambda_0 \delta(t) \rightarrow B = \Lambda_0 / A$
Governing Equations

• Extended Navier-Stokes Equation

\[
\frac{\rho}{\eta_0} \left( \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} \right) = -\nabla p + \mu_0 (\mathbf{M} \cdot \nabla) \mathbf{H} + 2\zeta \nabla \times \omega + (\lambda + \eta - \zeta) \nabla (\nabla \cdot \mathbf{v}) + (\zeta + \eta) \nabla^2 \mathbf{v} - \rho g \hat{i}_x
\]

Neglecting Inertia

Incompressible flow

• Boundary condition on \( \mathbf{v} \), \( \mathbf{v}(r = R_{wall}) = 0 \)

• Conservation of internal angular momentum

\[
\mathbf{J} \left( \frac{\partial \omega}{\partial t} + (\mathbf{v} \cdot \nabla) \omega \right) = \mu_0 (\mathbf{M} \times \mathbf{H}) + 2\zeta (\nabla \times \mathbf{v} - 2\omega) + (\lambda' + \eta') \nabla (\nabla \cdot \omega) + \eta' \nabla^2 \omega
\]

Neglecting Inertia

• Boundary condition on \( \omega \) unless \( \eta' = 0 \), \( \omega(r = R_{wall}) = 0 \)

\( \rho \) [kg/m\(^3\)] is the ferrofluid mass density, \( p \) [N/m\(^2\)] is the fluid pressure, \( \zeta \) [Ns/m\(^2\)] is the vortex viscosity, \( \eta \) [Ns/m\(^2\)] is the dynamic shear viscosity, \( \lambda \) [Ns/m\(^2\)] is the bulk viscosity, \( \omega \) [s\(^{-1}\)] is the spin velocity of the ferrofluid, \( \mathbf{v} \) is the velocity of the ferrofluid, \( \mathbf{J} \) [kg/m] is the moment of inertia density, \( \eta' \) [Ns] is the shear coefficient of spin viscosity and \( \lambda' \) [Ns] is the bulk coefficient of spin viscosity, \( \phi \) [%] is the magnetic particle volume fraction.
Magnetic Field Equations

• Maxwell’s equations for non-conducting fluid

\[
M = \text{Re}\left\{Me^{-j\Omega t}\right\}, \quad B = \text{Re}\left\{Be^{-j\Omega t}\right\}, \quad H = \text{Re}\left\{He^{-j\Omega t}\right\}
\]

\[
\nabla \cdot \mathbf{B} = 0 \rightarrow \frac{dB_x}{dx} = 0 \rightarrow B_x = \text{constant}
\]

\[
\nabla \times \mathbf{H} = 0 \rightarrow \frac{dH_z}{dx} = 0 \rightarrow H_z = \text{constant}
\]

\[
\mathbf{B} = \mu_0 \left( \mathbf{H} + \mathbf{M} \right)
\]

• Assumption

\[
M_{\text{eq}} = \chi H_{\text{fluid}}
\]

• Magnetic Relaxation Equation

\[
\frac{\partial \mathbf{M}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{M} - \omega \times \mathbf{M} + \frac{1}{\tau_{\text{eff}}} (\mathbf{M} - \mathbf{M}_0) = 0
\]

• Langevin Equation

\[
M_0 = M_s [\coth(a) - \frac{1}{a}], \quad a = \frac{\mu_0 H_0 M_d V_p}{kT}
\]

\[
\frac{1}{\tau_{\text{eff}}} = \frac{1}{\tau_B} + \frac{1}{\tau_N} \quad \tau_B = 3V_h \frac{\eta_0}{kT}, \quad \tau_N = \frac{1}{f_0} \exp\left(\frac{K_a V_p}{kT}\right)
\]

\(M_s [\text{Amps/m}]\) represents the saturation magnetization of the material, \(M_d [\text{Amps/m}]\) is the domain magnetization (446kA/m for magnetite), \(V_h\) is the hydrodynamic volume of the particle, \(V_p\) is the magnetic core volume per particle, \(T\) is the absolute temperature in Kelvin, \(k = 1.38 \times 10^{-23} [\text{J/K}]\) is Boltzmann’s constant, \(f_0 [1/\text{s}]\) is the characteristic frequency of the material and \(K_a\) is the anisotropy constant of the magnetic domains.
Substituting in Relaxation Equation

\[ j\Omega M_x - \omega_y M_z + \frac{M_x}{\tau} = \frac{\chi_0}{\tau} H_x \]

\[ j\Omega M_z + \omega_y M_x + \frac{M_z}{\tau} = \frac{\chi_0}{\tau} H_z \]

\[ B_x = \mu_0 (H_x + M_x) \rightarrow H_x = \frac{B_x}{\mu_0} - M_x \]

\[ B = \text{Re}[B_x i_x + B_z (x) i_z] e^{(-j\Omega t)} \]

\[ H = \text{Re}[H_x (x) i_x + H_z i_z] e^{(-j\Omega t)} \]

\[ M_x = \frac{\chi_0}{\left( \omega_y \tau \right)^2 + (j\Omega \tau + 1) \left( j\Omega \tau + 1 + \chi_0 \right)} \left[ H_z \left( \omega_y \tau \right) + (j\Omega \tau + 1) B_x / \mu_0 \right] \]

\[ M_z = \frac{\chi_0}{\left( \omega_y \tau \right)^2 + (j\Omega \tau + 1) \left( j\Omega \tau + 1 + \chi_0 \right)} \left[ H_z \left( j\Omega \tau + 1 + \chi_0 \right) - B_x \omega_y \tau / \mu_0 \right] \]
Force and Torque Densities

\[
\langle \mathbf{F} \rangle = \frac{\mu_0}{2} \text{Re} \left[ \left( \mathbf{M} \cdot \nabla \right) \mathbf{H}^* \right] \rightarrow \mathbf{F}_x = -\frac{d}{dx} \left( \frac{\mu_0}{4} |\mathbf{M}_x|^2 \right), \mathbf{F}_z = 0
\]

\[
\langle \mathbf{T} \rangle = \frac{\mu_0}{2} \text{Re} \left[ \mathbf{M} \times \mathbf{H}^* \right] \rightarrow \mathbf{T}_y = \frac{1}{2} \text{Re} \left[ \mathbf{M}_z \mathbf{B}_x^* - \mu_0 \mathbf{M}_x^* \left( \mathbf{M}_z + \mathbf{H}_z \right) \right]
\]

Linear and Angular Momentum Eqns

\[
0 = -\frac{\partial p'}{\partial z} + 2\zeta \frac{d\omega_y}{dx} + (\zeta + \eta) \frac{d^2v_z}{dx^2}
\]

\[
0 = \mathbf{T}_y - 2\zeta \left( \frac{dv_z}{dx} + 2\omega_y \right) + \eta' \frac{d^2\omega_y}{dx^2}
\]

\[
p' = p + \frac{\mu_0}{4} |\mathbf{M}_x|^2 + \rho g x
\]
Normalization and Substitution

\[ \tilde{\Omega} = \Omega \tau, \quad \tilde{H} = \frac{\hat{H}}{H_0}, \quad \tilde{M} = \frac{\hat{M}}{H_0}, \quad \tilde{B} = \frac{\hat{B}}{\mu_0 H_0}, \quad \tilde{x} = \frac{x}{d}, \quad \tilde{v}_z = \frac{v_z \tau}{d}, \quad \tilde{\omega}_y = \omega_y \tau, \]

\[ \tilde{T}_y = \frac{T_y}{\mu_0 H_0^2}, \quad \tilde{\eta} = \frac{2\eta}{\mu_0 H_0^2 \tau}, \quad \tilde{\eta}' = \frac{\eta'}{\mu_0 H_0^2 \tau d^2}, \quad \tilde{\zeta} = \frac{2\zeta}{\mu_0 H_0^2 \tau}, \quad \frac{\partial \tilde{p}'}{\partial \tilde{z}} = \frac{d}{\mu_0 H_0^2} \frac{\partial p'}{\partial z} \]

\[ \tilde{M}_x = \frac{\chi_0 \left[ \tilde{\omega}_y \tilde{H}_z + (j \tilde{\Omega} + 1) \tilde{B}_x \right]}{\left[ \tilde{\omega}_y^2 + (j \tilde{\Omega} + 1)(j \tilde{\Omega} + 1 + \chi_0) \right]} \]

\[ \tilde{M}_z = \frac{\chi_0 \left[ (j \tilde{\Omega} + 1 + \chi_0) \tilde{H}_z - \tilde{B}_x \tilde{\omega}_y \right]}{\left[ \tilde{\omega}_y^2 + (j \tilde{\Omega} + 1)(j \tilde{\Omega} + 1 + \chi_0) \right]} \]

\[ \langle \tilde{T}_y \rangle = \frac{1}{2} \text{Re} \left[ \tilde{M}_z \tilde{B}_x^* - \tilde{M}_x^* \left( \tilde{H}_z + \tilde{M}_z \right) \right] \]

\[ 0 = -\frac{\partial \tilde{p}'}{\partial \tilde{z}} + \tilde{\zeta} \left( \frac{d \tilde{\omega}_y}{d \tilde{x}} \right) + \frac{1}{2} \left( \tilde{\zeta} + \tilde{\eta} \right) \frac{d^2 \tilde{v}_z}{d \tilde{x}^2} \]

\[ < \tilde{T}_y > - \tilde{\zeta} \left( \frac{d \tilde{v}_z}{d \tilde{x}} + 2 \tilde{\omega}_y \right) + \tilde{\eta}' \frac{d^2 \tilde{\omega}_y}{d \tilde{x}^2} = 0 \]
Torque Density

- **Analytical Torque Density**

\[
\langle \tilde{T}_y \rangle = \frac{\chi_0}{2} \left[ -\tilde{\omega}_y \left| \tilde{B}_x \right|^2 \left( \tilde{\omega}_y^2 - \tilde{\Omega}^2 + 1 \right) + \left| \tilde{H}_z \right|^2 \left( \tilde{\omega}_y^2 - \tilde{\Omega}^2 + (1 + \chi_0)^2 \right) \right] \\
+ 2 \Re \left[ \chi_0 \left( \tilde{\omega}_y^2 - \tilde{\Omega}^2 \right) + i \tilde{\Omega} \left( \tilde{\omega}_y^2 - \tilde{\Omega}^2 - 1 - \chi_0 \right) \right] \left[ \tilde{H}_z \tilde{B}_x^* \right] \\
/ \left[ \left( \tilde{\omega}_y^2 + \tilde{\Omega}^2 + 1 + \chi_0 \right)^2 + \left( 2 + \chi_0 \right)^2 \tilde{\Omega}^2 \right].
\]

- **Small spin limit Torque Density**

\[
\lim_{\tilde{\omega}_y \ll 1} \langle \tilde{T}_y \rangle = \tilde{T}_0 + \alpha \tilde{\omega}_y
\]

\[
\tilde{T}_0 = -\frac{\chi_0 \Re \left[ \chi_0 \tilde{\Omega}^2 + j \tilde{\Omega} \left( \tilde{\Omega}^2 + 1 + \chi_0 \right) \right] \left[ \tilde{H}_z \tilde{B}_x^* \right]}{\left[ 1 + \chi_0 + \tilde{\Omega}^2 \right]^2 + \chi_0^2 \tilde{\Omega}^2}
\]

\[
\alpha = \frac{\chi_0}{2} \frac{\left| \tilde{B}_x \right|^2 \left( \tilde{\Omega}^2 - 1 \right) + \left| \tilde{H}_z \right|^2 \left( \tilde{\Omega}^2 - (1 + \chi_0)^2 \right)}{\left[ 1 + \chi_0 + \tilde{\Omega}^2 \right]^2 + \chi_0^2 \tilde{\Omega}^2}
\]
COMSOL Setup

- **Linear Momentum Equation**
  - 2D Incompressible Navier Stokes Module

\[
0 = -\frac{\partial \tilde{p}'}{\partial z} + \tilde{\zeta} \left( \frac{d \tilde{\omega}_y}{d\tilde{x}} \right) + \frac{1}{2} \left( \tilde{\zeta} + \tilde{\eta} \right) \frac{d^2 \tilde{v}_z}{d\tilde{x}^2}
\]

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<tr>
<th>COMSOL Subdomain quantities</th>
<th>Value</th>
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</thead>
<tbody>
<tr>
<td>( \rho )</td>
<td>0</td>
</tr>
<tr>
<td>( \eta )</td>
<td>( \frac{1}{2} (\tilde{\zeta} + \tilde{\eta}) )</td>
</tr>
<tr>
<td>( F_x, F_y )</td>
<td>( \tilde{\zeta} \left( \frac{d \tilde{\omega}_y}{d\tilde{x}} \right) )</td>
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**Inlet BC – Pressure, No viscous Stress**

\[ p_o = -\tilde{p}', \frac{\partial \tilde{p}'}{\partial z} > 0 \]

**Outlet BC, Normal Stress**

\( f_0 = 0 \)

**No Slip BC**

\( \tilde{v}_z = 0 \)
COMSOL Setup

• Angular Momentum Equation
  – General PDE Equation

\[
<\tilde{T}_y> - \tilde{\zeta} \left( \frac{d\tilde{v}_z}{dx} + 2\tilde{\omega}_y \right) + \tilde{\eta}' \frac{d^2\tilde{\omega}_y}{dx^2} = 0
\]

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<tr>
<td>(\Gamma)</td>
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<td>(F)</td>
<td>(&lt;\tilde{T}_y&gt;-\tilde{\zeta}\left(\frac{d\tilde{v}_z}{dx}+2\tilde{\omega}_y\right)+\tilde{\eta}'\frac{d^2\tilde{\omega}_y}{dx^2})</td>
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<td>(e_a,d_a)</td>
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Boundary Conditions

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<th>COMSOL Quantities</th>
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<tr>
<td>All walls (if (\tilde{\eta}' \neq 0))</td>
<td>Dirichlet boundary condition (R = -\tilde{\omega}_y), (G=0)</td>
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<tr>
<td>All walls (if (\tilde{\eta}' = 0))</td>
<td>Neumann boundary condition (G=0)</td>
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COMSOL Setup

- Magnetic Relaxation Equation
  - 2D Perpendicular Induction Currents, Vector Potential

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<td>( M )</td>
<td>( \frac{\chi_0 \left[ \tilde{\omega}_z \tilde{H}_z + (j\tilde{\Omega} + 1) \tilde{B}_x \right]}{\tilde{\omega}_y^2 + (j\tilde{\Omega} + 1)(j\tilde{\Omega} + 1 + \chi_0)} ), ( \frac{\chi_0 \left[ (j\tilde{\Omega} + 1 + \chi_0) \tilde{H}_z - \tilde{B}_x \tilde{\omega}_y \right]}{\tilde{\omega}_y^2 + (j\tilde{\Omega} + 1)(j\tilde{\Omega} + 1 + \chi_0)} )</td>
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Boundary Conditions | COMSOL Quantities |
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<td>All walls</td>
<td>( H_0 = \tilde{H}_z, \tilde{H}_x = \tilde{B}_x - \tilde{M}_x )</td>
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η'≠0 Results, Weak Rotating Fields

Parameters used – $\chi_0 = 1$, $\tilde{\eta} = 1$, $\tilde{\zeta} = 1$, $\frac{\partial \tilde{p}'}{\partial \tilde{z}} = 0.00001$, $\tilde{\Omega} = 1$, $\tilde{\eta}' = 0.01$
η’≠0 Results, Weak Rotating Fields

Parameters used – \( \chi_0 = 1, \tilde{\eta} = 1, \tilde{\zeta} = 1, \frac{\partial \tilde{p}'}{\partial \tilde{z}} = 0.00001, \tilde{\Omega} = 1, \tilde{\eta}' = 0.01 \)

η' = 0 Results, Strong Rotating Fields

Parameters used – \( \chi_0 = 1, \tilde{\eta} = 1, \tilde{z} = 1, \frac{\partial \tilde{p}'}{\partial \tilde{z}} = 0.00001, \tilde{\Omega} = 1, \tilde{\eta}' = 0 \)

η' = 0 Results, Strong Rotating Fields

Small Spin Velocity limit does not hold

Parameters used – \( \chi_0 = 1, \tilde{\eta} = 1, \tilde{\zeta} = 1, \frac{\partial \tilde{p}'}{\partial \tilde{\zeta}} = 0.00001, \tilde{\Omega} = 1, \tilde{\eta}' = 0 \)

“Kinks” for special parameters

Parameters used – \( \chi_0 = 1, \tilde{\eta} = \tilde{\zeta} = 0.0592, \frac{\partial \tilde{p}'}{\partial \tilde{z}} = 1, \tilde{\Omega} = 5, \tilde{\eta}' = 0 \)

Conclusions

- Ferrohydrodynamic flows are difficult to model
  - Coupling of five vector equations
    - Linear and angular momentum equations
    - Gauss’s law for magnetic flux density
    - Ampere’s law with no free current
    - Ferrofluid magnetic relaxation equation

- Solving the basic planar geometry ferrofluid pumping problem is valuable before moving to cylindrical and spherical geometries

- COMSOL gives identical results to prior software of choice - Mathematica